Report on Locational Price Risk Management

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for

The Electricity Authority[1]

Date: July 22, 2011

Version: 21

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Contents

Executive Summary

1. This document reviews the methodology for allocating rentals for the FTR design for the New Zealand electricity market as outlined in Schedule 14.6 to the Code.

2. The formulae in Schedule 14.6 are intended to ring fence rentals from certain parts of the network to form an FTR account. This will be used to fund the FTR products sold.

3. The intent of the formulae is to collect enough transmission rentals to cover a collection of balanced FTRs between the Benmore and Otahuhu grid exit points. The exact composition of this set of FTRs is not specified in the formula. The rentals to be added to the FTR account are computed for extreme flow patterns, yet to be determined.

4. The formulae use a decomposition of point-to-point flows by power transfer distribution factors to compute the rental contributions from each constraint.

5. We propose that the costs of fixed HVDC losses are not deducted from the FTR account.

6. We propose that some changes be made to computing transmission loss rentals to give upper bound (worst case) estimates of these.

7. We propose that some amendments are made to Schedule 14.6 to correct some typographical errors and reduce ambiguity in the definitions.

8. We note that the collection of rentals by the formulae in Schedule 14.6 is based on some approximations, the implications of which will need to be understood by the FTR provider and electricity market participants[2]. These are:

    a. Because of transmission losses, the estimated power transfer distribution factors are not exact. Therefore the rentals that are computed do not match exactly with the payments that must be made.

    b. When the HVDC sets the risk for spinning reserve, it is possible that there are insufficient transmission rentals to fund a set of simultaneously feasible FTRs.

[2] The discovery of these issues is not new. The Electricity Authority and other authors of Schedule 14.6 are aware of the approximations, and have discussed the consequences of them in accompanying documents.

Glossary of terms

Many of the terms here have standard definitions that can be found online (e.g. in Wikipedia)

Convex function: A function f is *convex* if the function of the average of two points is no more than the average of the function values at these points. The piecewise linear loss functions in SPD[3] are convex.

Concave function: A function f is *concave* if the function of the average of two points is no less than the average of the function values at these points.

Convex set: A set X is *convex* if the line segment joining any two points in X lies entirely in X.

Convex problem: An optimization problem that maximizes a concave function over a convex set, or minimizes a convex function over a convex set is called a *convex problem*. The problem solved by SPD is not a convex problem unless the energy balance constraints are expressed as ≥ constraints (see *free disposal*).

Congestion payment: The payout (or obligation if negative) of any financial instrument like a financial transmission right or flow-gate right.

Financial transmission right (FTR): A *financial transmission right* (FTR) from nodes *i* to node *j* is defined by a pair of numbers (u,v) that pays (or is obliged to pay if it is negative) a congestion payment of

$$(\pi_j v_j + \pi_i u_i)$$

where π_i is the *nodal price* at node *i*.[4]

Flow-gate right (FGR): A *flow-gate right* (FGR) on a line *k* in a given direction pays a congestion payment of the *shadow price* on the capacity constraint or zero (depending on the direction) for every unit of the right held.

Free disposal: A dispatch model allows *free disposal* if the flow constraints at any node are of the form

$$\sum flowin - \sum flowout \geq demand - generation$$

This means that energy can be freely disposed of at nodes if it exceeds demand. This requirement ensures convexity of the dispatch model when convex losses are present. In practice the constraint is nearly always satisfied as an equation (unless there are negative prices).[5]

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[3] Scheduling, Pricing and Dispatch (SPD) is the market clearing software in the New Zealand wholesale electricity market.

[4] An "option" FTR (u,v) can also be defined. It pays the holder max(0, $\pi_j v_j + \pi_i u_i$). We will focus in this report on obligation FTRs.

[5] For more discussion of this see Palma-Benhke, R., Philpott A.B., Jofre, A. and Cortes-Carmona, M. Modelling network-constrained economic dispatch problems, to appear in *Engineering Optimization*, 2010, downloadable from www.epoc.org.nz.

FTR (balanced): A *balanced financial transmission right* (BFTR) from node *i* to node *j* pays (or is obliged to pay if it is negative) a congestion payment of $(\pi_j - \pi_i)$ for every unit of the right held, so it is an FTR of the form (-1,1). In most cases we will be discussing balanced FTRs in this document and so the term FTR will mean a BFTR unless otherwise stated.

FTR (unbalanced): A *unbalanced financial transmission right* (UFTR) from nodes *i* to node *j* is defined by a pair (*u*,*v*) that pays (or is obliged to pay if it is negative) a congestion payment of $(\pi_j v_j + \pi_i u_i)$. Typically (*u*,*v*) will have the form $(-1-\alpha/2, 1-\alpha/2)$, where α is an estimate of the thermal losses of a unit of flow from *i* to *j*.

Generator rental: The generator rental for a generator *g* at node *i* when dispatched x, is the difference in the revenue earned by the generator and its cost $C_g(x)$[6] of delivering *x*. So its rental is $\pi_i x - C_g(x)$, where π_i is the nodal price at node *i*.

Nodal price: The energy spot price (denoted π_i for node *i*) delivered by SPD.

On-the-Day dispatch: The dispatch and prices that occur on the day that the (e.g. FTR) instrument uses to determine its congestion payment. We call this the *otd dispatch*[7].

On-the-Day grid: The grid that is used for the otd dispatch. Since this occurs in the future, this is not known at the time of the FTR auction, since there may be outages or changes in grid values.

Power Transfer Distribution Factor (PTDF): Also known as a *shift factor*, this is denoted D_{ki} and it gives the fraction of flow from node i to node 1 that would be transported along line *k*. D_{ki} is constant for constant marginal losses, but varies with dispatch when there are increasing marginal losses.

Reserve rental: The *reserve rental* for a reserve provider *r*, is the difference in the reserve revenue earned by the provider and its declared cost $B_r(y)$[8] of delivering reserve y. So its rental is $\rho y - B_r(y)$, where ρ is the reserve price.

Revenue adequacy: An allocation of instruments with congestion payments contingent on a future dispatch is said to be *revenue adequate* if the total congestion payment can be funded entirely from transmission rentals earned by the SCM in that dispatch.

Scheduling Pricing and Dispatch (SPD): The dispatch and pricing software used to clear the New Zealand wholesale electricity spot market.

Shadow price: The *shadow price* on a constraint in SPD (also known as a *dual variable*) is the nonnegative number that gives the increase in system cost that would be obtained by tightening the constraint by one

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[6] This is total cost, not marginal cost, computed to be the area under the offer stack.
[7] For clarity, this is the dispatch resulting from the final pricing run of SPD, using metered demand.
[8] This is total cost, computed to be the area under the provider's reserve offer stack.

unit. Shadow prices are returned automatically from SPD. To give the above interpretation all constraints must be expressed in SPD as

LHS ≥ RHS

if SPD is minimizing cost and

LHS ≤ RHS

if SPD is maximizing consumer and producer surplus.

Shift Factor: See PTDF above.

Simultaneous feasibility: A collection of (possibly unbalanced) FTRs is *simultaneously feasible* if they meet the all the transmission constraints, the branch security constraints, and mixed constraints of the otd grid. Simultaneous feasibility is a sufficient condition for revenue adequacy in convex dispatch problems when these constraints involve branch flow variables only. If they include terms relating to dispatched generation or reserve then the simultaneous feasibility test is invalid (see Appendix 2).

System clearing manager (SCM): The organisation responsible for determining final wholesale electricity prices in each trading period. These are determined by the SCM running SPD with final offers and metered loads.

Transmission loss: The thermal loss in energy in a transmission line. In SPD, these are represented by piecewise linear convex functions of branch flow.

Transmission rental: The transmission rental accrued by the SCM is the total difference in payments made by loads minus the total revenue paid to generators. In our definition we ignore payments from reserve setters and to reserve providers. The transmission rental is therefore $\sum_i \pi_i g_i(f)$ where $g_i(f)$ is the total flow into node i if the vector of line flows is f.[9]

[9] We demonstrate this in the technical appendix (Appendix 2) to this report.

Purpose of this document

The Electricity Authority released a consultation paper on 12 April 2011 containing proposed amendments to the Electricity Industry Participation Code 2010 for consultation with stakeholders in the electricity industry as part of its Locational Price Risk Management Project. Submissions on this consultation paper closed at 5pm on 12 May 2011.

The consultation paper included a number of items related to the revenue adequacy of Financial Transmission Rights (FTRs), including:

(a) draft Code (schedule 14.6 to part 14 of the draft Code) on the calculation of amount of loss and constraint excess to be paid into the FTR account;

(b) Appendix D: Explanation of payouts under obligation and option FTRs; and

(c) Appendix E: Determination of share of AC rentals used for funding inter-island FTR.

The terms of reference of this review are to determine whether the formulae used in Schedule 14.6 are correct. The formulae define rental streams that accrue to the system operator due to constraints in the dispatch software Schedule Price and Dispatch (SPD). The intention of the formulae in the schedule is to identify a proportion of the rentals from these constraints that can be used to provide revenue for a financial transmission right (FTR) between two points in the network. It is hoped that this design can be augmented to accommodate more FTR products at a later stage.

In this document we first review the formulae in (a), (b), and (c), and comment on their correctness.

We then discuss the methodology underlying Schedule 14.6 and comment on the overall approach used for funding FTRs. This section is written in the form of some questions that may arise about the approach, and some answers to these questions.

We then provide some general remarks on the proposed design, and suggest a possible direction in which the proposed system might evolve. This last section is more speculative than the preceding sections, and the statements made here should not be interpreted as firm recommendations.

The paper contains two appendices:

Appendix 1 gives a background to the methodology underlying financial transmission rights. Appendix 2 is a detailed mathematical appendix that can be consulted to see derivations of formulae and proofs of some of the assertions made in this report. Although “option” financial transmission rights are included in the proposed design, we restrict attention in our discussion to (standard) “obligation” financial transmission rights and their properties.

Remarks on overall Schedule 14.6 approach (Appendix E of consultation paper)

Appendix E of the April 2011 consultation paper describes the approach in Schedule 14.6 to ring fence the rental pool to enable some rentals to be retained for later instruments. The essence of this model is to describe the rental pool as accruing from the proportions of rentals from a restricted set of lines that are predetermined in the FTR auction. These lines come from so-called extreme-point balanced flows in the system. The FTRS that are sold are convex combinations of these extreme flows.

The major drawback in this approach is that extreme balanced flows are not feasible for a network with losses. So an FTR corresponding to an extreme balanced flow may not be revenue adequate.

Appendix E also attempts to separate HVDC flows from AC flows between Otahuhu and Haywards. If one ignores the effect of reserve, it should be possible to make this separation explicit by auctioning flow-gate rights (FGRs) on the HVDC to allocate these rentals, and then combining these with an FTR between Haywards and Otahuhu. The sales of FTRs between Otahuhu and Haywards can be constrained to be revenue adequate. This option is discussed in more detail in the last section of this report.

The formula in Appendix D of the April 2011 consultation paper (when amended to give the correct congestion payments to be paid to the holder of an FTR, as discussed later in this report) is based on the price difference between the nodes. Since this price difference reflects losses and reserve as well as line capacity constraints, there are implications for revenue adequacy.

The implications of not accounting for transmission losses when using balanced FTRs have been discussed at length in the consultation document and in submissions. What is less well understood is the effect of reserve on rentals. When the HVDC sets the instantaneous reserve risk, there may be an energy price difference between Haywards and Benmore nodes even though the line is not at its transmission limit. This means that a flow-gate right (FGR)[10] on this line will have zero congestion payment (since the shadow price is zero), but an FTR between Benmore and Haywards will have a nonzero congestion payment. It is possible therefore for a revenue shortfall to occur[11]. This is discussed in section 1.3 (Theorem 6) of Appendix 2 to this report.

[10] Refer to Appendix 1 for an explanation of a flow-gate right.

[11] The revenue shortfall from transmission rentals is not because the costs of supplying reserve are to be met from transmission rentals. It exists even if these are met from an external levy. Its source is the energy price difference between endpoints of the HVDC line that comes from the need to pay for extra reserve if we transport more power. This is reflected in energy price differences that result in a congestion payment, even though there may be no transmission rentals to fund it.

Schedule 14.6: Remarks by Section

In this section of the report, we go through Schedule 14.6 section by section, and identify any errors or ambiguities. This part of the report should be construed as a critique of the Schedule 14.6 document rather than a critique of the overall design.

Section 1: Purpose

We have no remarks to make on this section.

Section 2: Interpretation

This section provides definitions of many of the technical terms used. It is important that these are precise and unambiguous, and so we recommend that this section be carefully rewritten.

It is not clear in this section what is meant by a “constraint”. There appears to be an implicit assumption that all “constraints” are inequalities. In some circumstances an inequality constraint can satisfied as an equation but with a zero shadow price. According to the definition this is not “binding”. The authors should be clear that this interpretation is what they intend.

It would help to list all the generic types of constraints from SPD that are intended to be covered by this schedule.

The term “constraint price” is not defined in this interpretation. Can such a price be negative? If not, then it should explicitly be stated that prices are nonnegative.

Once prices are defined, “LHS” and “RHS” need some careful definition here. It is not stated what the “canonical form” of the constraint is. Dantzig[12] defines the canonical form of a system of linear programming equations with an initial basis of the identity, but we don’t think that this is what is intended here.

What is important is that each SPD constraint is written in the form

$$(LHS)_i \geq (RHS)_i$$

where $(LHS)_i$ is a linear function and $(RHS)_i$ is a (possibly negative) number[13]. The “constraint price” for this constraint can then be defined to be the corresponding shadow price for this constraint. This will be non-negative by convention. This is made clear by analysing the Lagrangian of the dispatch problem, which is done in detail in Appendix 2.

Section 3: Loss and constraint excess

The amount transferred to the FTR account is decreased by the instantaneous reserve costs charged to the HVDC owner. This will decrease the amount that is available to support revenue adequacy of FTR instruments. Even if these reserve costs are not deducted, the FTR

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[12] Dantzig, G.B. Linear Programming and Extensions, Princeton, 1963.
[13] Assuming the SPD is minimizing total dispatch and reserve cost.

congestion payments may exceed line rentals in circumstances where the HVDC is setting the risk (see Appendix 2).

Section 4: Calculation of loss and constraint excess

General remarks

The formulae in this section are written using the same notation used to describe SPD. In our view this is cumbersome, and can serve to obscure what is going on, but we can see that there are good reasons for using it here. Notwithstanding this necessity, the formulae appear to be written in the most economical form possible, condensing several cases into a single equation. This makes them difficult to interpret. Our recommendation is that different cases are treated separately, or explained in detail, so that they are easily open to scrutiny.

(1) "constraint prices" are not defined. See Section 2 above.

(2) *HVDC Rental*

The HVDC rental formula is consistent with how the HVDC is modelled in SPD. The first term ("abs()") defines the rental

$$\sum_{n(SI)} \pi_n g_n(f) + \sum_{n(NI)} \pi_n g_n(f)$$

as defined in Appendix 2. Taking the absolute value here is not required.

The HVDC fixed loss terms should not be included at all in the formula. In the dispatch solution these fixed losses can be thought of as extra load at the endpoints of each HVDC line. This will have an effect on the prices at these endpoints, but does not affect the set of feasible arc flows in the HVDC lines. Therefore the transmission rentals should not be reduced by the fixed-loss terms[14].

(3) *AC Rental*

"constraint price" is not defined here. See Section 2 above.

(4) *Security constraint rental*

"constraint price" is not defined here. See Section 2 above. It is not clear why security constraint rentals are taken to be the positive part. Can they be negative? This needs some explanation.

(5) *AC loss rental*

The design treats AC line losses as piecewise linear functions with losses incurred at the destination node. This is consistent with the SPD formulation.

When the AC line flow lies between two breakpoints the term

[14] To see why, imagine the fixed losses were enormous. We meet them in dispatch by importing from the AC network. The rentals on the HVDC line might be reduced to zero by the fixed losses, but we would still have to fund a nonzero FTR congestion payment.

$$0.5(ACLineLossFactor_{k,lowermarg} + ACLineLossFactor_{k,uppermarg})$$

becomes the slope α of the loss function at the optimal flow value. The rental for any fully-used loss tranche (j) with slope α_j and capacity u_j is then $(\alpha-\alpha_j)u_j\pi$, where π is the receiving end price. This formula is derived in Appendix 2.

The formula incorporates the term

$$0.5(ACLineLossFactor_{k,lowermarg} + ACLineLossFactor_{k,uppermarg}).$$

This is a heuristic rule to deal with the case when the AC line flow is exactly on a breakpoint of a loss tranche. At this point the loss factor is not well defined, and there is a nonuniqueness in determining the shadow price of the constraint. The heuristic rule averages the extreme values of the possible interval of shadow prices.

It is not clear why this heuristic is chosen. To ensure revenue adequacy in all circumstances (where SPD might choose the price at the upper end of this interval) it would be prudent to always choose the right-hand limit $ACLineLossFactor_{k,uppermarg}$ in the formula instead of the average.

Section 5: Process for determining capacities to be assigned

The proposal intends to test for simultaneous feasibility using a convex set defined by "extreme" points[15]. In fact these are not actually extreme in the technical sense of the word, but feasible points close to the boundary of the feasible set. Once a set of these points has been chosen, it is intended to sell FTRs that lie in the convex hull of this set.

Given a collection of FTRs, Section 5 determines what share of the transmission rentals each product should be allocated. We do not believe that this process is straightforward when losses and security constraints are included. As shown in Appendix 2 the PTDFs vary with the dispatch and so the allocation of rentals using *Assigned Capacity* will not exactly match the congestion payments from the FTR products.

The determination of the assigned capacity can be done at the time of the auction, but prior to otd dispatch. This would be achieved by using equation (1) and the PTDFs to determine what fraction of the shadow price on each constraint should be allocated to the holder of an FTR.

In 15 (a) a formula is given to determine upper bound on the maximum possible capacity that could be used by an FTR that is the convex combination of extreme flows[16]. This formula allows the SCM to allocate future rentals from the grid *prior* to the FTR auction. The formula is an upper bound as the actual FTRs that are to be sold are assumed to be unknown (while lying in some pre-determined convex set). It is not clear

[15] We recognize that this construction might not be the final methodology used by the FTR auctioneer, and we would support a more generic test for simultaneous feasibility being defined in Schedule 14.6.
[16] Similarly in part (b) it is the most negative value if the flow is opposite to the conventional direction of the branch.

how conservative this bound will be in practice, and some care should be taken in estimating how tight it is[17].

The subsection (16) relates to security and mixed constraints. In (16) the word "each" is repeated. Each security constraint in SPD has the form

$$-s^T f \geq -a, \quad [\mu \geq 0]$$

(with shadow price μ). The "weight" associated with security constraint *v* is not defined in (16) but if we assume that this is s_k then (1) becomes

$$\pi_j - \pi_i = \sum_k (\rho_k - \sigma_k + \mu s_k)(D_{ki} - D_{kj})$$

which is consistent with the formula in (16).

Some care is needed with allocating rentals from mixed constraints which can be quite general involving both branch flow and generation variables. Recall that simultaneous feasibility is a sufficient condition for revenue adequacy in convex dispatch problems when these constraints involve branch flow variables only. If they include terms relating to dispatched generation or reserve then the simultaneous feasibility test is invalid (see Appendix 2).

[17] In the last section of this report we discuss an alternative model in which the FTRs are ring fenced by scaling at the time of the FTR auction rather than by allocating proportions of rentals from specific parts of the grid. If the otd grid is known then this procedure ensures revenue adequacy, and does not risk assigning too much capacity to the FTR product which might occur if the bound underlying 15(a) is too loose.

Comments on overall approach to funding FTRs

The intention of the proposed design is to allocate proportions of line rentals from different lines to a rental stream for an FTR. The reason for this is to enable the rental stream for FTRs to be ring fenced in some way, so that the residual rentals are available for other hedging instruments at a later stage.

For a classical DC load-flow model without losses or additional constraints, one can use PTDFs to represent a financial transmission right by a portfolio of flow-gate rights[18]. It is then possible to directly compute the congestion payments from the flow-gate rights (defined by ρ_k and σ_k) and construct a weighted portfolio of congestion payments for the FTR. The PTDFs can be negative and positive, and so the portfolio can involve both short and long positions on flowgate rights, which are options.

The proposed approach outlined in Schedule 14.6 is similar to this decomposition, with the difference that it precludes short positions using the definition of AssignedCapacity. In this respect, the approach is intended to give an upper bound on the rental to be collected to fund an FTR.

This construction raises some questions when the network has transmission losses and other constraints (such as security constraints) affecting the line flows. These are:

Question 1:

Is the construction valid when there are line losses?

As shown in Appendix 1, the total transmission rentals when the vector of branch flows from the otd dispatch is *y* is defined by

$$\sum_i \pi_i g_i(y)$$

where $g_j(y)$ is the total flow from the network into node *i*. If marginal line losses are constant (so $g_j(y)$ is a linear function of *y*) then there are no loss rentals. In the absence of a binding transmission constraint on a line between two nodes, the congestion payment for an unbalanced FTR on these two nodes will be zero[19].

If line losses are piecewise linear convex or nonlinear (e.g. quadratic) then the construction of PTDFs is not exact. It becomes an approximation. When losses are not linear, one can compute PTDFs that provide the paths that flow from *i* to *j* travels along for the particular dispatch. This is done by solving the dispatch problem and identifying the transmission flows in each line. The PTDFs will vary with the dispatch.

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[18] See Section2 of Appendix 2 for a derivation of this decomposition.

[19] As shown in Appendix 2, for a single line (*i,j*) the optimal flow *y* will satisfy $-\pi_i\, g_i(y) = \pi_j\, g_j(y)$ which means that the transmission rental from this line is zero.

It follows that attempting to trace flows using PTDFs is not valid when losses are present. The approach proposed in this design therefore provides only an approximation of equation (1) above. This means that the congestion payments from an FTR between two nodes will not exactly match the congestion payments computed from the portfolio of FGRs assembled using lossless PTDFs.

The issues regarding losses have been discussed at length in submissions to the Electricity Authority, and in the consultation paper. Nevertheless, the proposed design is to offer balanced point-to-point FTRs. Underlying this decision is a belief that these are simpler instruments than unbalanced FTRs or single-node FTRs[20]. Simplicity improves participation, which improves the outcomes from auctions of transmission rights and secondary trading.

The importance of liquidity in FTR markets is not an issue for debate in this report. However, it is not clear that secondary trading of FTRs outside the auction is as welfare enhancing as one might initially suppose. This is because secondary trades will lead to FTRs being owned by those who can exploit them best, in particular agents whose market power can extract strategic value from the FTRs. Of course, an auction might lead to the same outcome eventually, and FTR holders can periodically offer them back to the auction whenever it is held, but who then acquires the FTR is open to much more scrutiny than bilateral trading.

An environment in which agents bid for FTRs to suit their own hedging needs, without seeking to trade them outside the auction, leads one to an auction design with a wide array of balanced and unbalanced FTRs (including loss support contracts) of different sorts that are constructed by and bid for by participants, and are allocated to maximize auction revenue.

When losses are present, balanced FTRs on their own do not satisfy the sufficient conditions for revenue adequacy (i.e. simultaneous feasibility in the lossy network). However as shown by a two-node example in the technical appendix, a number of balanced point-to-point FTRs might be revenue adequate if their volume is much smaller than the observed point-to-point flow on the day of dispatch. This feature makes revenue adequacy dependent on knowing the *"on-the-day"* (otd) dispatch, which is an undesirable feature.

The most effective way to deal with a potential lack of revenue adequacy from balanced FTRs is to sell loss support contracts at the same time as allocating FTRs. This can be done with no risk by the FTR auctioneer. An example of this is given in Appendix 1.

[20] A "single-node FTR", or "loss support contract" of amount s at node 1, say, is a particular form of unbalanced FTR in which the auctioneer sells an unbalanced FTR defined by the vector (-s,0,…,0). The details of this are discussed in Appendix 2.

Question 2:

Is the construction valid when there are other constraints on flows apart from those in the classical model?

The use of PTDFs in the classical case can be used to unpack the relationship between other constraints on transmission flows and differences in nodal prices. For example if there is a security constraint of the form

$$-s^T f \geq -a, \quad [\mu \geq 0]$$

(with shadow price μ) then (1) becomes

$$\pi_j - \pi_i = \sum_k (\rho_k - \sigma_k + \mu s_k)(D_{ki} - D_{kj})$$

If there are transmission losses then the validity of the construction depends on the validity of the PTDFs. Since these now depend on the dispatch, the formulae are no longer exact.

Recall that simultaneous feasibility is a sufficient condition for revenue adequacy in convex dispatch problems when these constraints involve branch flow variables only. So when there are constraints linking reserve requirements to HVDC branch flow, or constraints mixing generation and branch flow then revenue shortfalls may occur, even with simultaneous feasibility.

Question 3:

Is the construction valid for convex combinations?

The design alludes to a test for simultaneous feasibility using convex combinations of candidate point-to-point balanced FTRs. This is correct in the absence of losses. If there are losses, then balanced FTRs alone will never be simultaneously feasible, unless they are chosen in such a combination so as to have zero flow.

Question 4:

Do the formulae in schedule 14.6 correctly compute rentals?

The formulae in schedule 14.6 contain some errors that are listed on pages 9 and 10 of this report. These can be corrected to give formulae that compute transmission rentals accruing to different constraints under an optimal dispatch.

Question 5:

What proportion of the total transmission rental should be allocated to a point-to-point FTR?

This question really only makes sense under the assumption that the point-to-point FTR is simultaneously feasible for the constraints. If it is not, then there is a risk that the FTR will not be able to be funded from all the rentals let alone some fraction of them.

The proof of the revenue adequacy theorem (see Appendix 2) requires that the dispatch problem is convex. If there are convex losses, this means that the flow balance constraints are written as ≥ constraints, where the inequality means that we allow free disposal of power at the nodes. In most instances of the dispatch problem, the flow balance constraints are satisfied as equations at optimality, i.e. there is no actual disposal of power in the optimal dispatch. This only occurs when negative nodal prices arise from the optimal dispatch.

These observations imply that any FTR that corresponds to a set of flows that is feasible in the network with free disposal allowed will be simultaneously feasible, and therefore will satisfy the conditions of revenue adequacy.

A balanced point-to-point FTR will not, on its own, be simultaneously feasible in this sense. It needs to be combined with unbalanced FTRs (for example single-node or "spot" FTRs or loss support contracts).

Remarks on explanation of FTR payouts (Appendix D of consultation paper)

Appendix D gives an explanation of the payouts under obligation and option FTRs. This contains some errors that need correcting.

D 1.1 The explanation is very imprecise. The payout from an obligation FTR is presumed to be the congestion payment received by the holder of the instrument to be paid by the seller of the instrument. If this payout is negative, then the holder of the FTR pays the seller minus the payout.

The FTR has a *direction* from node *j* to node *k*. Given this direction the *sending end* is node *j* and the *receiving end* is node *k*. If this is the case then the congestion payment for a *Q* MW FTR held for *K* trading periods is

$$P = \sum_{t=1}^{t=K} (\pi_k(t) - \pi_j(t))Q/2$$

This differs from the explanation that states that PD is the sending price minus the receiving price.

D1.2 The same error in sign occurs in the payout description of the option FTR.

General remarks on the design

FTRs and FGRs

There has been much debate in the literature on the relative merits of flow gate rights (FGRs) as compared with financial transmission rights (FTRs). As shown in Appendix 2, in the classical case with no reserve or losses, any FTR can be decomposed into a portfolio of FGRs using PTDFs. If FGRs were readily available, then a market participant could in principle construct any FTR they wanted from an appropriate portfolio of FGRs. However participants who wished to hedge nodal price differences must calculate PTDFs to do so. The revenue adequacy of any bundle of FTRs would follow from the fact that they are all funded from shadow prices on constraints. There would also be no need to ring fence constraint rentals and allocate them to an FTR account.

A point-to-point FTR that is sold in an auction transfers the complexity of computing PTDFs to the auction mechanism. This means that participants can directly bid for FTRs rather than constructing synthetic ones from FGRs. The disadvantage is that the rights sold at auction must satisfy certain constraints to guarantee revenue adequacy. This also means that each auction must take account of FTRs that are already held. Secondary trading of FTRs is arguably more difficult than secondary trading of FGRs.

The relative merits of these two alternatives become less clear when the network is subject to losses. In this case PTDFs are not constant and depend on the dispatch. This poses a risk for participants who might wish to use these to construct synthetic FTRs from FGRs.

On the other hand, as shown in Appendix 1 and Appendix 2, even with losses, it is possible using single-node or other unbalanced FTRs to guarantee revenue adequacy of a set of FTRs, assuming the grid does not change, and assuming that there are no constraints mixing transmission flow and dispatch of energy and reserve.

In summary, auctioning FGRs to participants would have advantages over FTRs in the absence of losses. With losses, it appears that an FTR auction with simultaneous feasibility is the easiest mechanism to achieve revenue adequacy.

The use of extreme-point flows

As described in Schedule 14.6 part 5, the proposal uses a convex set defined by "extreme" points to compute the rentals to be added to the FTR account. These extreme FTR flow patterns are determined by the FTR manager, and may not actually be extreme in the technical sense of the word, but feasible FTR flow patterns close to the boundary of the feasible set.

Let us denote by G the convex hull of the extreme flow patterns. The rentals collected on the day according to Schedule 14.6 will correspond to at least those that would arise from an optimal dispatch with flows restricted to G. If we ignore the issues of reserve and losses, these rentals will be sufficient by the revenue adequacy theorem to fund any set of FTRs with flow patterns in G. The FTR manager when determining what FTRs to sell in the auction must then at least ensure that the FTRs sold lie in G.

This construction can be compared with using a forecast of the otd network with modifications to deal with planned outages and contingengies. The SPD constraints are available, and given forecast grid availability for the duration of the FTR, it is simple to test whether a set of injection patterns is feasible. As long as the process for determining the forecast otd network is clearly defined and agreed upon by market participants, there is little room for litigating the outcome, which is not the case if extreme FTR flow patterns are defined inexactly.

The drawback with using the forecast otd grid in the simultaneous feasibility test, is that it makes it difficult to ensure ex ante that some parts of the rental stream are ring fenced, and not all allocated to those seeking to hedge price differences between the hubs. This is what Schedule 14.6 aims to do using extreme flow patterns.

In the full grid, one could attempt to ring fence rentals using some fictitious FTRs between other nodes prior to the auction. For example, one might suppose some extant FTRs between New Plymouth and Otahuhu, and between Haywards and Tokaanu. These products can be included together in an FTR auction program with a very high bid price. However it should be acknowledged that doing this poses a risk for the auctioneer, should the otd dispatch produce flows in opposite directions, obliging the auctioneer to contribute funds to pay the other (fictitious) congestion payments[21].

An alternative approach scales the FTRs sold in the auction by some proportion $\beta<1$. This can be implemented in several ways.

1. clear the auction subject to simultaneous feasibility, and then deliver a fraction β of what was bid by each successful bidder;
2. decrease the congestion payments on a contract to a fixed proportion $\beta<1$ of the price difference between the nodes[22].
3. scale the network (i.e. grid line capacities, loss tranche breakpoints, and security constraints) by β before applying the simultaneous feasibility test.

The effect of any of these mechanisms is to decrease the congestion payment for Q balanced FTRs over K trading periods to be

$$\sum_{t=1}^{t=K} (\pi_k(t) - \pi_j(t)) Q\beta / 2 .$$

[21] For example, reserving fictitious FTR products for a counterflow might allow a greater amount of FTRs to be sold in the current auction (to be financed on the day of dispatch by the obligatory payments from the holders of the fictitious FTR products).

[22] We have not studied this approach for option FTRs, and note that scaling may not work for these, as their payoff functions are not linear.

The balance of the transmission rentals

$$\sum_{t=1}^{t=K}(\pi_k(t)-\pi_j(t))Q(1-\beta)/2$$

that is not allocated by the auction can then be reserved for other payments. Here β can be gradually increased over time as there is more demand for the FTR products. Of course agents should also be able to offer their FTRs to later auctions for sale, possibly at a profit. Thus the auctioneer can buy these and repackage them as hedging products for higher paying agents.

The next steps for FTRs

It is interesting to speculate on the effect that adding more hubs will have on the FTR design. The simplest addition would be a hub at Haywards. This would enable a separation of HVDC flows from AC flows in the FTR grid that is already explicitly recognized in Schedule 14.6. We think that this would present several advantages.

One advantage would be the ability to sell FGR products on the HVDC. A market for FGRs on the HVDC link between Benmore and Haywards is a simple extension to the current allocation of these rentals. The option nature of this instrument is well suited to the "tidal" nature of flows between North and South Islands, and removes the need to test simultaneous feasibility in different circumstances. It would also simplify any adjustments to the payoffs to account for situations where the HVDC is setting the reserve risk.

Price differences in the remaining AC part of the grid between Haywards and Otahuhu can be hedged using FTRs, with unbalanced FTRs such as loss-support contracts included to provide revenue adequacy. An (obligation) FTR between Benmore and Otahuhu can then be constructed using a short and long position on HVDC flow-gate rights and an FTR between Haywards and Otahuhu.

Appendix 1: Explanation of key concepts of FTRs

Financial transmission rights are used to hedge location risk. In a single-node pool market there is a single system marginal price resulting in no location risk. Because loads pay for energy at the marginal price, and generators are paid at the marginal price, in the absence of any constraints, the dispatch from a single node pool market does not earn any revenue for the system clearing manager (SCM). All payments from loads are used to compensate generators.

On the other hand, in a fully nodal market like the New Zealand system, with losses on transmission lines, thermal limits on the flow in these lines, and other constraints imposed for security or reliability reasons, the amount paid by loads may exceed the amount paid to generators. The difference in payments, which accrues to the SCM, is called a *transmission rental*.

In some circumstances, it is possible to unpack the total transmission rental in any trading period in terms of contributions from individual constraints. The classical dispatch model considers a DC-load flow network with the only constraints being thermal capacity constraints on the transmission lines[23]. In this setting, each node has a *nodal price*, giving the marginal cost of supplying an extra unit of energy at the node, and each capacity constraint has a *shadow price*, giving the increase in dispatch cost of decreasing the capacity by one unit. If the line is not at capacity then its shadow price is zero. In this setting, the rental accrued by each constraint is its capacity multiplied by its shadow price. (We need only consider lines at capacity to compute these rentals.)

The simplest case of a nodal market has two nodes *i* and *j* joined by a line *k* of capacity *K*, say. The flow in the line *k* from *i* to *j* is denoted f_k and it can be negative (i.e. flow from *j* to *i*). This flow is subject to two constraints:

$$f_k \geq -K, \quad [\sigma_k \geq 0]$$

$$-f_k \geq -K, \quad [\rho_k \geq 0]$$

where we define in brackets the corresponding shadow prices σ_k and ρ_k. We write these as ≥ constraints to ensure nonnegative shadow prices[24]. The shadow prices can be interpreted as the *increase* in the cost of the optimal dispatch when K is made one unit smaller. (Equivalently they give the decrease in system cost when K is made one unit larger[25].) With the sign convention above we have

$$\pi_j - \pi_i = \rho_k - \sigma_k$$

[23] In contrast to the New Zealand electricity market, the classical model ignores losses.

[24] We assume here that SPD is minimizing system cost. This is the default for a problem to be solved by CPLEX, the optimization engine that computes the optimal dispatch. If SPD is maximizing consumer surplus then constraints need to be expressed in ≤ form.

[25] This assertion is true if the optimal cost is smooth as a function of K at the optimal solution. If this is not the case then the shadow price is not uniquely defined and can be any value in an interval.

where π_i = nodal price at node *i.*

The nodal prices will be the same unless the line is at capacity *K* either in the direction from *i* to *j* when $\rho_k>0$, $\sigma_k=0$, or in the direction from *j* to *i* when $\rho_k=0$, $\sigma_k>0$. In this setting, the rental earned by the SCM is either

$$(\pi_j - \pi_i)K = \rho_k K$$

or

$$(\pi_j - \pi_i)(-K) = \sigma_k K.$$

The model becomes more complicated when the transmission network has loops (but we still assume no losses or constraints apart from thermal limits). In this case one needs to account for the effect of line impedance on the dispatch flows. One way of modelling the effect of impedance is to use *Power Transfer Distribution Factors* (PTDFs) or *shift* factors. These provide the paths in the network along which injected power will flow from source to destination.

Each line *k* and node *i* has a PTDF D_{ki}. The PTDF is computed using an arbitrarily chosen node called the *swing bus*. From now on we will choose the swing bus to be node 1. The PTDF D_{ki} then gives the flow that would occur on line *k* if a single megawatt were injected at node *i* and extracted at the swing bus (node 1).

It is possible to compute PTDF values for every pair (k,i). We provide an example and details of how to do this in Appendix 2. If a unit of flow is injected at node *i* and then used at node *j*, then the flow in line *k* resulting from this flow is

$$f_k = D_{ki} - D_{kj}.$$

We now have the following formula for nodal price differences:

$$\pi_j - \pi_i = \sum_k \lambda_k (D_{ki} - D_{kj}) \qquad (1)$$

where π_i = nodal price at node *i*, and

$\lambda_k = \rho_k - \sigma_k$.

Equation (1) (which is derived in the technical appendix) expresses the fundamental relationship between flow-gate rights and financial transmission rights.

A *flow-gate right* (FGR) on a line *k* in a given direction pays a congestion payment of ρ_k or σ_k (depending on the direction) for every unit of the right held.

A *financial transmission right* (FTR) from node *i* to node *j* pays (or is obliged to pay if it is negative) a congestion payment of $(\pi_j - \pi_i)$ for every unit of the right held.

Thus in a nodal market without losses or security constraints it is possible to hedge price differences between nodes *i* and *j* by using either an FTR product that pays (π_j - π_i) or a weighted portfolio of FGRs on lines *k*, each with weight D_{ki} - D_{kj} .

The essential difference is that an FGR is an *option* to receive the nonnegative shadow price whereas an FTR in its simplest form carries an *obligation* to pay the congestion payment to the SCM if it is negative. We note that *option* FTRs can also be constructed, and are included in the proposed design. An option FTR from i to j pays max(0,π_j - π_i).

In the two-node example, the FTR of one unit from *i* to *j* would correspond to holding a FGR of one unit on the line in the direction from i to j and being short of a FGR of one unit in the reverse direction. So the payment on the instrument will be ρ_k if the line is constrained from *i* to *j* and -σ_k if the line is constrained from *j* to *i*. The option FTR in this case would be equivalent to the corresponding FGR.

Loss-support contracts

If the network has line losses, and participants bid for balanced FTRs then the FTR auctioneer can purchase single-node FTRs in the auction to compensate for the losses that are not accounted for in the balanced FTR. We illustrate this several examples in a two-node network that consider both fixed marginal losses and quadratic losses. Similar examples can be constructed with piecewise linear losses.

Consider the network in Figure 1. Suppose that participant A at node 2 is hedging purchases of power from node 1 by bidding $a/MW for a balanced FTR between nodes 1 and 2 of amount 100MW.

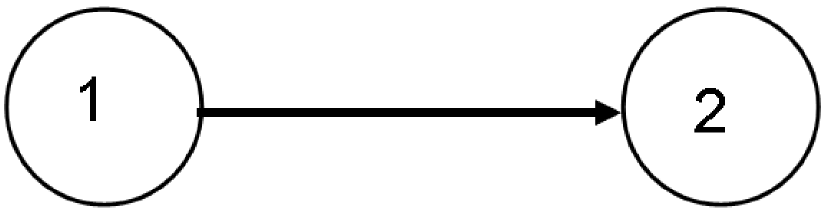


Figure 1

Example 1: Suppose the line has a constant marginal loss of 20%. Because 20MW of the 100MW of sent power will be lost, the auctioneer seeks offers of contracts for difference at node 2 of 20MW. Suppose that participant B offers this to the auction at $b/MW. This means that if the nodal price π_2 is above b, then the auctioneer is paid 20(π_2-b) and if it is less then the auctioneer pays 20(b-π_2). If 100a-20b>0, then the auctioneer sells the FTR and buys the contract. With the contract in place the auctioneer on the day will receive a payment of 20π_2.

On the day of dispatch, suppose that the price at node 1 is $100 and that there is 10 MW of flow in the line. Then the price at node 2 is $125 = $100/0.8. The transmission rental is

R_t = $125*8 – 100*10 = 0

The balanced FTR payment is ($125-$100)*100 = $2500.

The value of the contracted power is $125*20=$2500 which is enough to meet the congestion payment of the balanced FTR.

Example 2: Suppose the line has a quadratic loss function $T(f)=0.0025f^2$. A wants to hedge a purchase of 100MW, so they bid for a balanced FTR of 100MW. This gives lost power = T(100) = 25MW. The auctioneer seeks offers of contracts for difference at node 2 of 25MW. Suppose that participant B offers this to the auction at $b/MW. This means that if the nodal price π_2 is above b, then the auctioneer is paid $25(\pi_2-b)$ and if it is less then the auctioneer pays $25(b-\pi_2)$. If 100a-25b>0, then the auctioneer sells the FTR and buys the contract. With the contract in place the auctioneer on the day will receive a payment of $25\pi_2$.

On the day of dispatch, suppose that the price π_1 at node 1 is $100 and that there is only 10 MW of flow in the line. This gives a loss of T(10)=0.25MW. The price difference is given by

$$\pi_2(1-T'(f))= \pi_1$$

$$\pi_2(1-0.005*10)= \pi_1$$

So π_2=$105.26

The transmission rental is

R_t = $105.2632*9.75 – 100*10 = $26.3158

The balanced FTR payment is ($105.2632-$100)*100 = $526.3158.

The value of the contracted power is $105.26*25=$2631.58 which is enough to meet the congestion payment of the balanced FTR.

Example 3: Suppose the line has a quadratic loss function $T(f)=0.0025f^2$, and suppose that the price π_1 at node 1 is $100 and that there is 100 MW of flow in the line. This gives a loss of T(100)=25 MW. The price difference is given by

$$\pi_2(1-T'(f))= \pi_1$$

$$\pi_2(1-0.005*100)= \pi_1$$

So π_2=$200.00.

The transmission rental is

R_t = $200*75 – 100*100 = $5000.

The balanced FTR payment is ($200-$100)*100 = $10000.

The value of the contracted power is $200*25=$5000 which is enough with R_t to meet the congestion payment of the balanced FTR.

Example 4: Suppose the line has a quadratic loss function $T(f)=0.0025f^2$, and suppose that the price π_1 at node 1 is \$100 and that there is 150 MW of flow in the line. This gives a loss of T(150)=56.25 MW. The contracted loss is still 25 MW. The flow received at node 2 is then 93.75 MW. The price difference is given by

$$\pi_2(1-T'(f))= \pi_1$$

$$\pi_2(1-0.005*150)= \pi_1$$

So π_2=\$400.00.

The transmission rental is

R_t = \$400*93.75 – 150*100 = \$22500.

The balanced FTR payment is (\$400-\$100)*100 = \$30000.

The value of the contracted power is \$400*25=\$10000 which is enough with R_t to meet the congestion payment of the balanced FTR.

Appendix 2: On revenue adequacy in electricity pool markets with line losses and reserve constraints

Andy Philpott

1 Introduction

In this appendix we discuss the funding of *financial transmission rights* (FTRs) from rentals accrued by the Independent System Operator (the ISO) in a wholesale pool electricity market. Our purpose in this paper is to investigate the theory underlying revenue adequacy of these instruments in markets that are cleared by convex programming models. We focus on standard nodal pool markets with reserve-constrained dispatch and convex line losses, and investigate the implications of these.

A well-known result proved by Hogan states that revenue adequacy is guaranteed by the FTRs being *simultaneously feasible* for the network constraints, in the sense that they can be dispatched through the network without exceeding capacities. This result was first proved by Hogan [2] for lossless networks, extended to quadratic losses by Bushnell and Stoft [1], and further generalised to smooth nonlinear constraints by Hogan [3]. The result is a general consequence of the convexity of the dispatch problem and a separation between arc flows and dispatch. If these are linked through reserve-constrained dispatch then the revenue adequacy problem becomes more complicated as we discuss below.

The appendix is laid out as follows. In this introductory section we review the basic theory of revenue adequacy for convex pool markets, and prove several theoretical results that illustrate the role that reserve-constrained dispatch plays in ensuring revenue adequacy. In section 2, we consider the classical case where we ignore the effect of reserve and derive a formula for congestion payments for FTRs in the lossless case that expresses these as apportionments of constraint shadow prices to holders of FTRs. In section 3 we examine the effect of transmission losses. For ease of exposition, the analysis is carried out for quadratic losses and a single line, but in section 4 we show how the piecewise linear case can be analysed in a similar way. In section 5 we outline the theory of power transfer distribution factors, and show how these are computed. Fixed distribution factors are difficult to obtain with even constant marginal losses, because the loss amount at each node depends on the direction of flow that is not known a priori. Section 6 gives some examples for a simple meshed network that shows how lossless distribution factors might be poor estimates of the true power distribution in the case where there are quadratic losses.

1.1 Models and notation

We formulate the general dispatch problem as the following convex optimization problem in a transmission network with n nodes and m lines.

$$
\begin{array}{lllll}
\text{P:} & \text{minimize} & \sum_i \sum_{j\in O(i)} c_j x_j + \sum_i \sum_{j\in R(i)} b_j y_j & & \\
& \text{subject to} & g_i(f) + \sum_{j\in O(i)} x_j - z_i & = \; d_i, & i = 1,2,\ldots,n \\
& & \sum_i \sum_{j\in R(i)} y_j - w_1 & = \; q^\top x, & \\
& & \sum_i \sum_{j\in R(i)} y_j - w_2 & = \; s^\top f, & \\
& & f & \in \; F, & \\
& & (x,y) & \in \; X, & \\
& & z_i & \geq \; 0, & i = 1,2,\ldots,n \\
& & w_i & \geq \; 0, & i = 1,2.
\end{array}
$$

Here d_i is the demand at node i, and the first set of constraints represent conservation of flow at the nodes. We assume here that the problems associated with negative prices in situations of low demand do not occur, and so that at optimality the flow balance constraints are satisfied as equations. This means that we can represent the flow balance equations as inequalities, without losing any physical realism. The use of inequalites here is an important issue, as it affects revenue adequacy in an important way when we discuss losses.

In the formulation P, x_j is the level of dispatch of tranche $j \in O(i)$ where $O(i)$ is the set of tranches offered at node i, and each tranche $j \in O(i)$ is offered at price c_j. We also include a variable y representing ancillary services that are cleared at the same time as energy. For example, these could be spinning reserve or interruptible load dispatch that is offered as a set of tranches $j \in R(i)$ at node i at prices b_j. By $(x, y) \in X$ we denote all constraints on generation and ancillary services that do not involve line flows. So these might involve ramping, generation capacities, reserve limits from dispatched generation, and reserve requirements from risk-setting generation units.

A specific feature of our model is the link between reserve and the dispatch, as represented by the constraints $\sum_i \sum_{j \in R(i)} y_j \geq \max\{q^\top x, s^\top f\}$. The right-hand side of this constraint is the reserve requirement which is determined by the largest single risk, and at optimality is $\max\{q^\top x, s^\top f\}$. In this expression q might be a vector that selects the generation level of a particular unit, or s might be a vector that selects a particular line flow. In practice this maximum might be taken over many different collections of units and transmission lines, but without loss of generality we assume a simple comparison between generation and line flow will determine the reserve requirement. For simplicity we also assume only one constraint of this form, but in practice there might be several to handle for example fast and slow reserve, or reserve in different zones. We assume that all reserve payments corresponding to cleared bids b_j are met from revenues that do not accrue from constraint rentals. For example, the risk-setting generator might contribute to reserve payments.

The arc flows are measured by a vector f that must lie in some convex set F. Here the components of f can be either positive or negative (if moving in the opposite direction from an arbitrarily chosen direction for each link). The set F models any constraints that are satisfied by the arc flows alone. This includes loop-flow constraints that originate from Kirchhoff's Laws in the DC-load flow approximation, or security constraints imposed on branch flows to enable a quick response in times of outage (see e.g. [4]).

The function $g_i(f)$ is a general concave function giving the amount of power flow entering node i when the link flows are f. To define this in different settings we let

$$\begin{aligned} \mathcal{F}(i) &= \{k : k \text{ is an arc } \textit{from } i \text{ to some other node}\} \\ \mathcal{T}(i) &= \{k : k \text{ is an arc } \textit{to } i \text{ from some other node}\} \end{aligned}$$

In a lossless DC load-flow network we set

$$\begin{aligned} g_i(f) &= \sum_{k \in \mathcal{F}(i)} -f_k + \sum_{k \in \mathcal{T}(i)} f_k \\ &= -e_i^\top A f \end{aligned}$$

where e_i is the ith unit vector, and A is the $n \times m$ node-arc incidence matrix of the network with components

$$a_{ik} = \begin{cases} 1, & k \in \mathcal{F}(i), \\ -1, & k \in \mathcal{T}(i), \\ 0, & \text{otherwise.} \end{cases}$$

The most common approximation with losses defines $g_i(f)$ to be a concave quadratic function, meaning that branch losses are a quadratic function of power flow incurred equally at each end of the line. Formally,

$$g_i(f) = \sum_{k \in \mathcal{F}(i)} \left(-f_k - \frac{r_k}{2} f_k^2 \right) + \sum_{k \in \mathcal{T}(i)} \left(f_k - \frac{r_k}{2} f_k^2 \right) \tag{1}$$

where r_k is a constant loss coefficient for each line k.

The symmetric formulation (1) assumes that the flow is measured at the midpoint of the line. An alternative formulation assumes that y_k is measured when leaving a node. We could then set

$$g_i(f) = \sum_{k \in \mathcal{F}(i)} -f_k + \sum_{k \in \mathcal{T}(i)} \left(f_k - r_k f_k^2 \right)$$

which has the disadvantage that the transmission loss is not invariant to a simple change of sign in link flow. Thus, in most of the theory that follows we shall adopt the symmetric formulation (1).

In practice, a linear programming representation of P is often used. This treats branch flows as differences of nonnegative variables in directed arcs, and losses using convex piecewise linear functions, which makes $g_i(f)$ a concave piecewise linear function that subtracts the line losses of the flow at the receiving end of the line. We discuss this formulation and its relationship to revenue adequacy in detail in section 4 below.

1.2 Revenue adequacy theorems

Now suppose that $(x^*, y^*, f^*, z^*, w^*)$ solves P and that P satisfies a constraint qualification (e.g. P has a feasible solution with $z_i > 0$, $i = 1, 2, \ldots, n$, $w_i > 0$, $i = 1, 2$). Since P is a convex program it satisfies the Lagrangian Duality Theorem (see e.g [5]). This states that there is a set of optimal Lagrange multipliers π (the nodal prices for the optimal dispatch) and ρ_1 and ρ_2 (giving the reserve price) such that $(x^*, y^*, f^*, z^*, w^*)$ minimizes the Lagrangian

$$\begin{aligned}\mathcal{L}(x, y, f, z, w) \quad = \quad & \sum_i \sum_{j\in O(i)} c_j x_j + \sum_i \sum_{j\in R(i)} b_j y_j + \\ & \sum_i \pi_i (d_i - g_i(f) - \sum_{j\in O(i)} x_j + z_i) \\ & +\rho_1 (q^\top x + w_1 - \sum_i \sum_{j\in R(i)} y_j) \\ & +\rho_2 (s^\top f + w_2 - \sum_i \sum_{j\in R(i)} y_j)\end{aligned}$$

over $(x, y) \in X$, $z \geq 0$, $w \geq 0$, $f \in F$.

The Lagrange multipliers are used as prices in the pool market. This means that the ISO collects $\sum_i \pi_i d_i$ from all the loads, and pays $\sum_i \pi_i \sum_{j\in O(i)} x_j^*$ in total to the generators. Because of transmission constraints, transmission losses, and other constraints that may be binding, $\sum_i \pi_i d_i \geq \sum_i \pi_i \sum_{j\in O(i)} x_j^*$. We call the difference between these payments the *transmission rental*

$$R_t = \sum_i \pi_i d_i - \sum_i \pi_i \sum_{j\in O(i)} x_j^*.$$

The generators of energy also earn a *generator rental*

$$R_g = \sum_i \pi_i \sum_{j\in O(i)} x_j^* - \sum_i \sum_{j\in O(i)} c_j x_j^*,$$

and suppliers of reserve earn a *reserve rental*

$$R_r = \rho \sum_i \sum_{j\in R(i)} y_j^* - \sum_i \sum_{j\in R(i)} b_j y_j^*,$$

where $\rho = \rho_1$ (since $w_1^* = 0 < w_2^*$) if $q^\top x^* > s^\top y^*$ and $\rho = \rho_2$ (since $w_2^* = 0 < w_1^*$) if $q^\top x^* < s^\top y^*$.

An interesting special case occurs when $q^\top x^* = s^\top y^*$, which can happen quite often as the larger term (which sets the risk) is reduced to minimize the reserve cost until it meets the other term. In this case we have $w_1^* = 0 = w_2^*$ which does not uniquely determine ρ_1 and ρ_2. In mathematical terms the optimal value function is not smooth, and so the subdifferential is not a singleton. In practice, the optimization software makes an arbitrary choice of subgradient. A similar degeneracy occurs when the optimal dispatch is at a generation tranche boundary, and the marginal energy price can take any value in an interval.

It is easy to see that as long as $R_t \geq 0$, then there is enough income collected from loads to pay generators for their dispatch. Whether the dispatch problem yields enough revenue needed to meet reserve payments is not as straightforward. The total revenue earned from loads $\sum_i \pi_i d_i$ is in general not sufficient to fund the cost of reserve and energy $\sum_i \sum_{j\in O(i)} c_j x_j^* + \sum_i \sum_{j\in R(i)} b_j y_j^*$, and so it will not be sufficient to cover this cost as well as $R_g + R_r$. To see this, imagine a single node system in which all reserve is met from interruptible load and where all generators have the same marginal cost. Here $R_g = R_t = 0$. The payment to interruptible load therefore cannot be recovered from rentals.

We assume therefore that the income to pay suppliers of reserve comes from a levy on units that are creating the risk. Collectively they pay $\rho \sum_i \sum_{j\in O(i)} y_j^*$ which is enough to cover payment to reserve providers. We assume that the reserve rental is then allocated directly to reserve providers and is not available to fund congestion payments for FTRs.

If one takes out the settlement of the reserve then one can see that $R_t + R_g$ is the total rent from energy supply. The amount R_g accrues to generators, and R_t accrues to the ISO. We are interested in whether R_t is large enough to cover congestion payments for FTRs. Observe that R_t includes the effect that reserve has on the dispatch and transmission, but does not include reserve rentals R_r as defined above. Moreover R_t includes all the effects of losses, voltage support constraints, security constraints etc., as long as there are no constraints explicitly linking dispatch or reserve to line flow apart from those appearing in P.

This construction allows us to derive several lemmas.

Lemma 1 *$\pi_i \geq 0$ and ρ_1, $\rho_2 \geq 0$, and $\pi_i(g_i(f^*) + \sum_{j\in O(i)} x_j^* - d_i) = 0$, $\rho_1(\sum_i \sum_{j\in R(i)} y_j^* - q^\top x^*) = 0$, $\rho_2(\sum_i \sum_{j\in R(i)} y_j^* - s^\top f^*) = 0$.*

Proof. If $\pi_i < 0$, then for any feasible z, $\pi_i z_i \leq 0$, and so $\mathcal{L}(x, y, f, z, w)$ is unbounded below over $z \geq 0$. Similarly $\rho_1 < 0$ or $\rho_2 < 0$ gives an unbounded Lagrangian. Since $\mathcal{L}(x, y, f, z, w)$ has a minimum by the Lagrangian Duality Theorem, it follows that $\pi_i \geq 0$ and ρ_1, $\rho_2 \geq 0$. Now the minimizing choice of z_i^* must make $\pi_i z_i^* = 0$, , $i = 1, 2, \ldots, n$, and $\rho_i w_i^* = 0$, $i = 1, 2$, similarly. But since by feasibility

$$z_i^* = g_i(f^*) + \sum_{j \in O(i)} x_j^* - d_i,$$

and

$$w_1^* = \sum_i \sum_{j \in R(i)} y_j^* - q^\top x^*,$$

$$w_2^* = \sum_i \sum_{j \in R(i)} y_j^* - s^\top f^*,$$

this entails the result. ■

Lemma 2 *If $s^\top f^* < q^\top x^*$ then $\rho_2 = 0$. If $s^\top f^* > q^\top x^*$ then $\rho_1 = 0$.*

Proof. If $s^\top f^* < q^\top x^*$ then

$$\sum_i \sum_{j \in R(i)} y_j^* - s^\top f^* > \sum_i \sum_{j \in R(i)} y_j^* - q^\top x^* \geq 0$$

whence $\rho_2 = 0$ follows by virtue of Lemma 1. If $s^\top f^* > q^\top x^*$ then $\rho_1 = 0$ by a similar argument. ■

Lemma 3 $R_t = \sum_i \pi_i g_i(f^*)$

Proof.

$$R_t = \sum_i \pi_i d_i - \sum_i \pi_i \sum_{j \in O(i)} x_j^* = \sum_i \pi_i g_i(f^*)$$

by virtue of Lemma 1. ■

Lemma 4 *For every $x \in X$, $f \in F$, $R_t \geq \sum_i \pi_i g_i(f) + \rho_2(s^\top f^* - s^\top f)$*

Proof. Since $(x^*, y^*, f^*, z^*, w^*)$ minimizes $\mathcal{L}(x, y, f, z, w)$, f^* should be chosen in F to minimize

$$\begin{aligned}
\mathcal{L}(x^*, y^*, f, z^*, w^*) \;=\; & \sum_i \sum_{j\in O(i)} c_j x_j^* + \sum_i \sum_{j\in R(i)} b_j y_j^* + \\
& \sum_i \pi_i (d_i - g_i(f) - \sum_{j\in O(i)} x_j^* + z_i^*) \\
& + \rho_1 (q^\top x^* - \sum_i \sum_{j\in R(i)} y_j^* + w^*) \\
& + \rho_2 (s^\top f - \sum_i \sum_{j\in R(i)} y_j^* + w^*)
\end{aligned}$$

So, for all $f \in F$,

$$-\sum_i \pi_i g_i(f^*) + \rho_2 s^\top f^* \le -\sum_i \pi_i g_i(f) + \rho_2 s^\top f$$

Thus

$$\begin{aligned}
R_t \;&=\; \sum_i \pi_i g_i(f^*) \\
&\ge\; \sum_i \pi_i g_i(f) + \rho_2 (s^\top f^* - s^\top f).
\end{aligned}$$

■

Lemma 5 *Transmission rentals are sufficient to cover reserve from arc flows, i.e.* $R_t \ge \rho_2 s^\top f^*$

Proof. By Lemma 4 for every $x \in X$, $f \in F$,

$$R_t \ge \sum_i \pi_i g_i(f) + \rho_2 (s^\top f^* - s^\top f)$$

which gives the result if we choose $f = 0 \in F$. ■

Lemma 5 appears to contradict an earlier statement we made that $R_t + R_g$ might be insufficient to cover reserve payments. Certainly if $s^\top f^* > q^\top x^*$ then R_t alone will be enough to cover reserve payments. On the other hand, if $s^\top f^* \le q^\top x^*$ then a reserve payment $\rho_1 r^\top x^*$ may be required to support payment to reserve providers (e.g. from interruptible load), even if there is no transmission network and $R_g = 0$. This is why we have assumed that reserve payments should be funded from an external levy, and not deducted from transmission rentals R_t. However, even under this assumption, we observe that transmission rentals may be insufficient to fund congestion payments to an FTR.

1.3 Financial transmission right auctions

We now proceed to prove a revenue adequacy theorem for financial transmission rights in systems with reserve and losses. We define a financial transmission right (FTR) to be a vector $h(\alpha)$ that pays the holder the congestion payment $\sum_i \pi_i h_i(\alpha)$ for a given one-hour period. If $\sum_i \pi_i h_i(\alpha) < 0$ then this is an obligation on the holder to pay $-\sum_i \pi_i h_i(\alpha)$. If $\sum_i h_i(\alpha) = 0$ then the FTR is said to be *balanced*. Otherwise the FTR is said to be unbalanced. This definition is quite general and admits *single-node contracts* at some node n where $h_i(\alpha) = 0, i \neq n$. Such single-node contracts can be interpreted as a contract for differences of quantity $h_n(\alpha)$ at node n, in the sense that the holder might pay a fixed price p for such a contract in return for $\sum_i \pi_i h_i(\alpha) = \pi_n h_n(\alpha)$.

The financial transmisson rights are sold in an auction in which each agent $\alpha = 1, 2, \ldots, A$ offers a fixed amount $B(\alpha)$ tied to a vector $h(\alpha)$ in return for the price rental stream $\sum_i \pi_i h_i(\alpha)$ which he receives if the auction fully accepts his bid. Observe that single node contracts allow the FTR auctioneer to sell balanced FTRs in networks with losses, as long as they purchase at the same time some single-node contracts to cover the payments due to energy losses. The simultaneous feasibility conditions of the auction are:

$$\text{SF:} \quad \begin{array}{rcll} g_i(v) - z_i & = & \sum_\alpha h_i(\alpha), & i = 1, 2, \ldots, n, \\ z_i & \geq & 0, & i = 1, 2, \ldots, n, \\ v & \in & F. & \end{array}$$

The simultaneous feasibility conditions provide revenue adequacy in situations when the flows in transmission lines are predicted to be small enough so that $s^\top f^* \leq q^\top x^*$.

Theorem 6 *If the dispatch satisfies $s^\top f^* < q^\top x^*$ then R_t is sufficient to fund the congestion payments to any feasible solution to SF.*

Proof. Observe that SF implies

$$g_i(v) - z_i = \sum_\alpha h_i(\alpha)$$

The congestion payment to be made to this collection of FTR bids is

$$\begin{aligned}\sum_i \pi_i \sum_\alpha h_i(\alpha) &= \sum_i \pi_i(g_i(v) - z_i) \\ &\leq \sum_i \pi_i g_i(v),\end{aligned}$$

since $z_i \geq 0$, and $\pi_i \geq 0$. Now by virtue of Lemma 4, since $v \in F$,

$$\begin{aligned}\sum_i \pi_i g_i(v) &\leq R_t - \rho_2(s^\top f^* - s^\top f) \\ &= R_t\end{aligned}$$

because $s^\top f^* < q^\top x^*$ implies $\rho_2 = 0$ by Lemma 2. ■

1. If $s^\top f^* > q^\top x^*$, then the transmission rental might not be enough to cover the FTR congestion payments. In this case, we might require an extra $\rho_2(s^\top f - s^\top f^*)$ to guarantee revenue adequacy. This statement is independent of how reserve payments are funded. We are not assuming that the ISO is providing any funds from R_t to meet reserve payments. We have assumed that $\rho \sum_i \sum_{j \in O(i)} y_j^*$ is covered by a levy from outside the dispatch.

2. The potential shortfall in revenue is $\rho_2(s^\top f - s^\top f^*)$. To ensure revenue adequacy we could impose the constraint

   $$s^\top f \leq s^\top f^*$$

   on the FTR auction, but since we do not know the value $s^\top f^*$ a priori, it is not easy to do this.

3. An alternative is to impose obligations on FTR holders to reduce the congestion payments by $\rho s^\top f_\alpha$ where ρ is the reserve price, and f_α the contribution to $s^\top v$ from $h(\alpha)$. This is relatively easy to do for lossless FTRs but becomes more difficult when losses are present.

2 The classical case

In order to understand the relationships between *flowgate* rights and financial transmission rights, we review the classical theory. Here there is no reserve or line losses, so

$$\begin{aligned} g_i(f) &= \sum_{k\in\mathcal{F}(i)} -f_k + \sum_{k\in\mathcal{T}(i)} f_k \\ &= -e_i^\top Af \end{aligned}$$

Kirchhoff's laws for the DC-load flow model imply that the line flows f must satisfy

$$LCf = 0$$

where L is a $p \times m$ matrix with components +1, -1 and 0 representing p oriented loops in the network. The entries in L are

$$l_{jk} = \begin{cases} -1, & \text{if flow } f_k \text{ is in the opposite direction to the orientation of loop } j, \\ 1, & \text{if flow } f_k \text{ is in the same direction to the orientation of loop } j, \\ 0, & \text{if loop } j \text{ does not contain branch } k. \end{cases}$$

The matrix C is an $m \times m$ diagonal matrix of branch reactances.

An alternative model for DC-load flow uses voltage phase angles θ to represent Kirchhoff's voltage law, which is (for a connected network)

$$f_k = \frac{\theta_i - \theta_j}{C_k},\ \theta \in \mathbb{R}^n,\ \theta_1 = 0,$$

where we assign node 1 to be the *swing* bus. In matrix terms these equations are

$$f = C^{-1}A^\top\theta,\ \theta_1 = 0.$$

We can then eliminate the variables θ_i by writing

$$\begin{aligned} LCf &= LCC^{-1}A^\top\theta \\ &= LA^\top\theta \\ &= 0 \end{aligned}$$

where the last equation follows from the fact that the rows of L and A are orthogonal.

This gives the following problem:

$$
\begin{array}{llcll}
\text{P:} \quad \text{minimize} & \sum_i \sum_{j\in O(i)} c_j x_j & & & \\
\quad \text{subject to} & -e_i^\top A f + \sum_{j\in O(i)} x_j - z_i & = & d_i, & i=1,2,\dots,n \\
& -f & \geq & -K, & [\rho] \\
& f & \geq & -K, & [\sigma] \\
& LCf & = & 0 & [\tau] \\
& x & \in & X, & \\
& z_i & \geq & 0, & i=1,2,\dots,n.
\end{array}
$$

The Lagrangian is

$$
\begin{aligned}
\mathcal{L}(x,f,z) \;=\; & \sum_i \sum_{j\in O(i)} c_j x_j + \sum_i \pi_i (d_i + e_i^\top A f - \sum_{j\in O(i)} x_j + z_i) \\
& + \sum_k \rho_k(-K_k + f_k + w_{1k}) + \sum_k \sigma_k(-K_k - f_k + w_{2k}) \\
& + \tau^\top LCf
\end{aligned}
$$

which is minimized over $f \in F$. This gives

$$
\left(\pi^\top A\right)_k + \rho_k - \sigma_k + \left(\tau^\top LC\right)_k = 0
$$

$$
-\left(\pi^\top A\right)_k = \rho_k - \sigma_k + \left(\tau^\top LC\right)_k
$$

Let $\bar{A}$ be A with the first row deleted. If we postmultiply this equation on both sides by $D_{ki} - D_{kj}$ where $D = C^{-1}\bar{A}^\top(\bar{A}C^{-1}\bar{A}^\top)^{-1}$ and sum then we obtain

$$
\begin{aligned}
-\sum_k \left(\pi^\top A\right)_k (D_{ki} - D_{kj}) \;&=\; \sum_k (\rho_k - \sigma_k)(D_{ki} - D_{kj}) \\
&\quad + \sum_k \left(\tau^\top LC\right)_k D_{ki} - \sum_k \left(\tau^\top LC\right)_k D_{kj} \\
&=\; \sum_k (\rho_k - \sigma_k)(D_{ki} - D_{kj})
\end{aligned}
$$

since the rows of L are orthogonal to the rows of $\bar{A}$. Now

$$
\begin{aligned}
-\sum_k \left(\pi^\top A\right)_k (D_{ki} - D_{kj}) \;&=\; \sum_k \left(\pi^\top A\right)_k D_{kj} - \sum_k \left(\pi^\top A\right)_k D_{ki} \\
&=\; \pi^\top e_j - \pi^\top e_i
\end{aligned}
$$

This gives

$$\pi_j - \pi_i = \sum_k (\rho_k - \sigma_k)(D_{ki} - D_{kj}). \tag{2}$$

The equation (2) expresses an important relationship between financial transmission rights and flowgate rights. A *flowgate* right of 1 MW on a transmission link pays a congestion payment of ρ_k if the link is congested (at capacity) in the forward direction and σ_k if the link is congested in the reverse-flow direction. This payment is always non-negative and so the flowgate right is an option. An FTR of 1 MW on the same link in the direction from i to j pays $\pi_j - \pi_i$, which can be negative (giving an obligation). The equation (2) shows that a financial transmission right is equivalent to a portfolio of flow-gate rights. In the simplest case of a single line where the distribution factors are -1 and +1 the FTR is equivalent to being long in a flowgate right from i to j and short in the flowgate right in the reverse direction.

3 Smooth transmission losses

In this section we analyse congestion payments when transmission losses are smooth functions of the flow. We restrict attention to the case where reserve is not affecting the flows on any line, but is determined by constraints of the form $(x, y) \in X$. This gives the following dispatch problem.

$$\begin{array}{llrcl} \text{P:} & \text{minimize} & \sum_i \sum_{j\in O(i)} c_j x_j + \sum_i \sum_{j\in R(i)} b_j y_j & & \\ & \text{subject to} & g_i(f) + \sum_{j\in O(i)} x_j - z_i & = & d_i, \quad i = 1, 2, \ldots, n \\ & & f & \in & F, \\ & & (x, y) & \in & X, \\ & & z_i & \geq & 0, \quad i = 1, 2, \ldots, n. \end{array}$$

Since there are no reserve constraints affecting transmission, Theorem 6 becomes

Theorem 7 *The transmission rentals R_t are sufficient to fund the congestion payments to any feasible solution to SF.*

This version of the revenue adequacy theorem is the one we shall be using from now on. Recall that

$$R_t = \sum_i \pi_i g_i(f^*)$$

so the transmission rentals can be computed entirely from the nodal prices and transmission flows f^*. In this form, it is not necessary to list all the constraints and their corresponding multipliers.

To see how this is done, suppose that the constraints $f \in F$ take the form of thermal transmission limits:

$$\begin{aligned} -f - w_1 &= -K, && [\lambda] \\ f - w_2 &= -K, && [\mu] \\ w_i &\geq 0, && i = 1, 2. \end{aligned}$$

Here we have listed the corresponding Lagrange multipliers. The Lagrangian becomes

$$\begin{aligned} \mathcal{L}(x^*, y^*, f, z^*, w^*) &= \sum_i \sum_{j \in O(i)} c_j x_j^* + \sum_i \sum_{j \in R(i)} b_j y_j^* + \\ & \sum_i \pi_i (d_i - g_i(f) - \sum_{j \in O(i)} x_j^* + z_i^*) \\ & + \lambda^\top (-K + f + w_1^*) \\ & + \mu^\top (-K - f + w_2^*) \end{aligned}$$

which is minimized over all f. This means that

$$\frac{\partial}{\partial f_k} \left((\lambda - \mu)^\top f - \sum_i \pi_i g_i(f) \right) = 0$$

so

$$\lambda_k - \mu_k = \frac{\partial}{\partial f_k} \sum_i \pi_i g_i(f). \tag{3}$$

If there are no losses, then $g_i(f) = -e_i^\top A f$. For k, the line from i to j, (3) becomes

$$\begin{aligned} \lambda_k - \mu_k &= -\frac{\partial}{\partial f_k} \pi^\top A f \\ &= \pi_j - \pi_i \end{aligned}$$

If on the other hand we have quadratic losses, then

$$g_i(f) = \sum_{k \in \mathcal{F}(i)} \left(-f_k - \frac{r_k}{2} f_k^2 \right) + \sum_{k \in \mathcal{T}(i)} \left(f_k - \frac{r_k}{2} f_k^2 \right).$$

Now, if k is the line from i to j then (3) becomes

$$\begin{aligned}\lambda_k - \mu_k &= \frac{\partial}{\partial f_k}(\pi_i \left(-f_k - \frac{r_k}{2} f_k^2\right) + \pi_j \left(f_k - \frac{r_k}{2} f_k^2\right)) \\ &= \pi_j(1 - r_k f_k) - \pi_i(1 + r_k f_k)\end{aligned}$$

Multiplying by f_k gives

$$\begin{aligned}(\lambda_k - \mu_k) f_k &= (\pi_j - \pi_i) f_k - \pi_j r_k f_k^2 - \pi_i r_k f_k^2 \\ &= \pi_j \left(f_k - \frac{r_k}{2} f_k^2\right) - \pi_j \frac{r_k}{2} f_k^2 \\ &\quad + \pi_i \left(-f_k - \frac{r_k}{2} f_k^2\right) - \pi_i \frac{r_k}{2} f_k^2\end{aligned}$$

If the line k is not at a thermal limit, then $w_1^* > 0$ and $w_2^* > 0$, so $\lambda_k = \mu_k = 0$. This then gives the relationship between the prices π_i and π_j defined by

$$\frac{\partial}{\partial f_k} \sum_i \pi_i g_i(f) = 0$$

whence

$$\pi_j(1 - r_k f_k) = \pi_i(1 + r_k f_k).$$

The transmission rental on the line is then

$$\begin{aligned}R_t(k) &= \pi_j \left(f_k - \frac{r_k}{2} f_k^2\right) + \pi_i \left(-f_k - \frac{r_k}{2} f_k^2\right) \\ &= \pi_i \frac{r_k}{2} f_k^2 + \pi_j \frac{r_k}{2} f_k^2\end{aligned}$$

which is the same as the market value of losses.

If the line k is at its upper thermal limit, then we might have $\lambda_k > 0$, giving

$$\frac{\partial}{\partial f_k} \sum_i \pi_i g_i(f) > 0.$$

With quadratic losses this becomes

$$\pi_j(1 - r_k f_k) = \pi_i(1 + r_k f_k) + \lambda_k$$

Thus

$$\begin{aligned}\lambda_k f_k &= (\pi_j - \pi_i) f_k - \pi_j r_k f_k^2 - \pi_i r_k f_k^2 \\ &= \pi_j \left(f_k - \frac{r_k}{2} f_k^2\right) - \pi_j \frac{r_k}{2} f_k^2 \\ &\quad + \pi_i \left(-f_k - \frac{r_k}{2} f_k^2\right) - \pi_i \frac{r_k}{2} f_k^2\end{aligned}$$

The transmission rental earned from a line $k = (i, j)$ with $f_k = K$ is then

$$\begin{aligned} R_t(k) &= \pi_i \left(-f_k - \frac{r_k}{2} f_k^2\right) + \pi_j \left(f_k - \frac{r_k}{2} f_k^2\right) \\ &= \pi_i \frac{r_k}{2} f_k^2 + \pi_j \frac{r_k}{2} f_k^2 + \lambda_k f_k \end{aligned}$$

The first two terms represent the loss rental (which is the same with quadratic losses as the market value of lost power), and the last term is the rental due to the capacity limit. This analysis has assumed that the variable f_k (measured halfway along the line k) is constrained by thermal limits (rather than the sent flow $f_k + \frac{r_k}{2} f_k^2$).

In summary, in the absence of losses,

$$\lambda_k - \mu_k = \pi_j - \pi_i$$

and so the transmission rental earned from a line $k = (i, j)$ with $f_k = K$ is

$$(\pi_j - \pi_i) f_k = \lambda_k f_k = \lambda_k K.$$

With quadratic losses,

$$\pi_j (1 - r_k f_k) = \pi_i (1 + r_k f_k) + \lambda_k$$

so the transmission rental earned from a line $k = (i, j)$ with $f_k = K$ is

$$R_t(k) = \pi_i \frac{r_k}{2} f_k^2 + \pi_j \frac{r_k}{2} f_k^2 + \lambda_k f_k$$

We give the formula for piecewise linear losses in section 4.

3.1 Balanced FTRs

Consider a balanced FTR of value $q > 0$. If we assume that losses are strictly positive, then this is never simultaneously feasible for P even if we ignore reserve, and allow free disposal of electricity at the destination node.That is

$$\begin{aligned} \text{SF:} \quad g_1(v) - z_1 &= -q, \\ g_i(v) - z_i &= 0, \quad i = 2, \ldots, n-1, \\ g_n(v) - z_n &= q \\ z_i &\geq 0, \quad i = 1, 2, \ldots, n, \\ v &\in F \end{aligned}$$

has no solution unless $\sum_i g_i(v) = 0$. This can be seen by adding the constraints of SF to give

$$\sum_i g_i(v) = \sum_i z_i \geq 0$$

and observing that with strictly positive losses $\sum_i g_i(v) \leq 0$.

This shows that balanced (i.e. lossless) FTRs and their convex combinations will never be simultaneously feasible in networks with losses. However, simultaneous feasibility is a *sufficient* condition for revenue adequacy. A set of FTRs might satisfy revenue adequacy even if SF is infeasible. To understand this, we examine how revenue adequacy might fail with losses in the following example.

3.2 Example: two node network

Consider a two-node network with a large capacity line from node 1 to 2 with differentiable transmission functions $g_1(f)$ and $g_2(f)$, where f is the flow from node 1 to node 2. Suppose that the offer price at node 1 is π_1. When the optimal flow in the line is f (assumed to be in the interior of F) the price π_2 at node 2 satisfies the stationarity condition

$$\pi_2 g_2'(f) + \pi_1 g_1'(f) = 0.$$

Thus

$$\pi_2 = \pi_1 \left(\frac{-g_1'(f)}{g_2'(f)} \right).$$

The congestion payment for a balanced FTR $(-q, q)$ is then

$$(\pi_2 - \pi_1)q = \pi_1 \left(\frac{-g_1'(f)}{g_2'(f)} - 1 \right) q.$$

The rental received from the actual dispatch is

$$\pi_1 g_1(f) + \pi_2 g_2(f) = \pi_1 \left(g_1(f) + g_2(f) \frac{-g_1'(f)}{g_2'(f)} \right)$$

Revenue adequacy requires

$$\left(g_1(f) + g_2(f) \frac{-g_1'(f)}{g_2'(f)} \right) \geq \left(\frac{-g_1'(f)}{g_2'(f)} - 1 \right) q$$

or

$$g_2'(f)g_1(f) - g_2(f)g_1'(f) \geq (-g_1'(f) - g_2'(f))\, q$$

Now suppose that losses are quadratic functions of line flow. Then

$$\begin{aligned} g_1(f) &= -f - \frac{r}{2}f^2, \qquad g_1'(f) = -1 - rf, \\ g_2(f) &= f - \frac{r}{2}f^2, \qquad g_2'(f) = 1 - rf, \end{aligned}$$

so

$$g_2'(f)g_1(f) - g_2(f)g_1'(f) \geq (-g_1'(f) - g_2'(f))\, q$$

which simplifies to

$$(1 - rf)(-f - \frac{r}{2}f^2) - (f - \frac{r}{2}f^2)(-1 - rf) \geq 2rfq$$

or

$$q \leq \frac{f}{2}.$$

The implication is that a balanced FTR will be revenue adequate as long as the flow on the line in the dispatch is at least twice the amount of financial transmission rights that we sell on the line (in the direction of flow). Otherwise the revenue shortfall is

$$\pi_1 \frac{rf}{1 - rf}(2q - f).$$

If h is close to f, as one would expect from a lossless version of SF, then the revenue shortfall will be approximately

$$\pi_1 \frac{rf^2}{1 - rf} = \pi_1 \frac{rf^2}{1 - rf}$$

which is the market value of the total amount of lost energy (when evaluated at price corresponding to a fictitious node at the line's midpoint).

3.3 Unbalanced FTRs

As shown above, no convex combination of balanced FTRs will ever be feasible for SF, unless they result in flows v with $\sum_i g_i(v) = 0$. This is highly unlikely to happen. So we now turn our attention to unbalanced FTRs. Recall that a financial transmission right $h(\alpha)$ is unbalanced if $\sum_i h_i(\alpha) \neq 0$. This allows for single-node contracts of quantity $h_n(\alpha)$ at node n.

The example in the previous section raises the question as to what form of unbalanced FTR we could sell to restore revenue adequacy. Suppose we sell $(-(1+\alpha)q, (1-\alpha)q)$. The congestion payment for the unbalanced FTR is then

$$((1-\alpha)\pi_2 - (1+\alpha)\pi_1)q = \pi_1 \left(\frac{-g_1'(f)}{g_2'(f)}(1-\alpha) - (1+\alpha) \right) q.$$

We seek the minimum value of α while requiring revenue adequacy, so

$$\left(g_1(f) + g_2(f)\frac{-g_1'(f)}{g_2'(f)} \right) \geq \left(\frac{-g_1'(f)}{g_2'(f)}(1-\alpha) - (1+\alpha) \right) q.$$

For symmetric quadratic losses, to ensure revenue adequacy we require

$$(-f - \frac{r}{2}f^2) + (f - \frac{r}{2}f^2)\frac{1+rf}{1-rf} \geq \left(\frac{1+rf}{1-rf}(1-\alpha) - (1+\alpha) \right) q$$

which gives

$$rf^2 - 2qrf + 2q\alpha \geq 0,$$

entailing that

$$\alpha \geq \frac{rf(2q-f)}{2q}. \tag{4}$$

Observe that if we choose $q = f$, then we need $\alpha \geq \frac{rf}{2}$. This means that we can retain revenue adequacy by selling unbalanced FTRs $(-(1+\alpha)f, (1-\alpha)f)$, as long as α exceeds half the marginal loss at the optimal dispatch. Observe that $(-(1+\frac{rf}{2})f, (1-\frac{rf}{2})f)$ is (just) feasible for SF. Thus being feasible for SF is both necessary and sufficient when $q = f$ as one would expect from the proof of Theorem 6.

We could however choose q and α together so that (4) was satisfied. For balanced FTRs we would restrict q to half the expected line flow. For unbalanced FTRs we could increase the number of FTRs allocated by reducing the congestion payments using α.

4 Piecewise linear losses

4.1 Loss rentals

In the New Zealand dispatch model, SPD, line flows are assumed to be positive and each branch has two such variables, one in each direction. The losses are piecewise linear and accrued at the destination node. Thus if k is the line from i to j, the rental contribution from line k is

$$R_t(k) = -f_k\pi_i + \pi_j\left(f_k - T(f_k)\right)$$

where $T(f_k)$ is the loss from flow f_k. Suppose that

$$0 < f_k < K$$

in the dispatch solution. Then

$$T(f_k) = \alpha_{k1}u_{k1} + \alpha_{k2}u_{k2} + \ldots + \alpha_{ks}f_{ks}$$

where

$$\begin{array}{c} f_k = u_{k1} + u_{k2} + \ldots u_{k,s-1} + f_{ks} \\ 0 \le f_{ks} < u_{ks}. \end{array}$$

If the line is not at its thermal limit, and no other constraints involving f are binding then

$$\begin{aligned} \pi_i &= \pi_j\left(\frac{-g_2'(f)}{g_1'(f)}\right) \\ &= \pi_j\frac{-1 + T'(f_k)}{-1} \\ \pi_i &= \pi_j(1 - \alpha_{ks}) \end{aligned}$$

Thus

$$\begin{aligned} R_t(k) &= -f_k\pi_j(1-\alpha_{ks}) + \pi_j\left(f_k - (\alpha_{k1}u_{k1} + \alpha_{k2}u_{k2} + \ldots + \alpha_{ks}f_{ks})\right) \\ &= \pi_j(f_k\alpha_{ks} - (\alpha_{k1}u_{k1} + \alpha_{k2}u_{k2} + \ldots + \alpha_{ks}f_{ks})) \\ &= \pi_j((u_{k1} + u_{k2} + \ldots u_{k,s-1} + f_{ks})\,\alpha_{ks} \\ &\quad -(\alpha_{k1}u_{k1} + \alpha_{k2}u_{k2} + \ldots + \alpha_{ks}f_{ks})) \\ &= \pi_j((\alpha_{ks} - \alpha_{k1})u_{k1} + (\alpha_{ks} - \alpha_{k2})u_{k2} + \ldots + (\alpha_{ks} - \alpha_{k,s-1})u_{k,s-1}) \end{aligned}$$

If $f_{ks} = 0$, then the question arises, what is the price difference between π_i and π_j? Since g is not differentiable at this point there is a interval of prices that will support this optimal solution. SPD selects one of the extreme values depending on how CPLEX deals with degenerate solutions. For definiteness, one could assume that on average SPD will return shadow prices that average their extreme values, i.e.

$$\pi_i = \pi_j(1 - \frac{(\alpha_{k,s-1} + \alpha_{ks})}{2}),$$

or that SPD always chooses the largest price difference so

$$\pi_i = \pi_j(1 - \alpha_{ks}).$$

In the former case we obtain

$$\begin{aligned} R_t(k) &= -f_k\pi_j(1 - \frac{(\alpha_{k,s-1} + \alpha_{ks})}{2}) + \pi_j\left(f_k - (\alpha_{k1}u_{k1} + \alpha_{k2}u_{k2} + \ldots + \alpha_{ks}f_{ks})\right) \\ &= \pi_j(f_k\frac{(\alpha_{k,s-1} + \alpha_{ks})}{2} - (\alpha_{k1}u_{k1} + \alpha_{k2}u_{k2} + \ldots + \alpha_{ks}f_{ks})) \\ &= \pi_j((u_{k1} + u_{k2} + \ldots u_{k,s-1} + f_{ks})\,\frac{(\alpha_{k,s-1} + \alpha_{ks})}{2} \\ &\quad -(\alpha_{k1}u_{k1} + \alpha_{k2}u_{k2} + \ldots + \alpha_{ks}f_{ks})) \\ &= \pi_j\{(\frac{(\alpha_{k,s-1} + \alpha_{ks})}{2} - \alpha_{k1})u_{k1} + (\frac{(\alpha_{k,s-1} + \alpha_{ks})}{2} - \alpha_{k2})u_{k2} + \\ &\quad \ldots + (\frac{(\alpha_{k,s-1} + \alpha_{ks})}{2} - \alpha_{k,s-1})u_{k,s-1}\}. \end{aligned}$$

In the latter case we have

$$\begin{aligned} R_t(k) &= -f_k\pi_j(1 - \alpha_{ks}) + \pi_j\left(f_k - (\alpha_{k1}u_{k1} + \alpha_{k2}u_{k2} + \ldots + \alpha_{ks}f_{ks})\right) \\ &= \pi_j(f_k\alpha_{ks} - (\alpha_{k1}u_{k1} + \alpha_{k2}u_{k2} + \ldots + \alpha_{ks}f_{ks})) \\ &= \pi_j((u_{k1} + u_{k2} + \ldots u_{k,s-1} + f_{ks})\,\alpha_{ks} \\ &\quad -(\alpha_{k1}u_{k1} + \alpha_{k2}u_{k2} + \ldots + \alpha_{ks}f_{ks})) \\ &= \pi_j\{(\alpha_{ks} - \alpha_{k1})u_{k1} + (\alpha_{ks} - \alpha_{k2})u_{k2} + \ldots + (\alpha_{ks} - \alpha_{k,s-1})u_{k,s-1}\}. \end{aligned}$$

If the line is at its upper thermal limit (still while no other constraints involving f are binding) then (3) becomes

$$\begin{aligned} \lambda_k &= \frac{\partial}{\partial f_k}(\pi_i\,(-f_k) + \pi_j\,(f_k - T\,(f_k))) \\ &= \pi_j(1 - T'(f_k)) - \pi_i \end{aligned}$$

so

$$\pi_i = \pi_j(1 - \alpha_{ks}) - \lambda_k$$

$$\begin{aligned} R_t(k) &= -f_k\left(\pi_j(1-\alpha_{ks}) - \lambda_k\right) + \pi_j\left(f_k - (\alpha_{k1}u_{k1} + \alpha_{k2}u_{k2} + \ldots + \alpha_{ks}f_{ks})\right) \\ &= \pi_j(f_k\alpha_{ks} - (\alpha_{k1}u_{k1} + \alpha_{k2}u_{k2} + \ldots + \alpha_{ks}f_{ks})) + \lambda_k f_k \\ &= \pi_j((u_{k1} + u_{k2} + \ldots u_{k,s-1} + f_{ks})\,\alpha_{ks} \\ &\quad -(\alpha_{k1}u_{k1} + \alpha_{k2}u_{k2} + \ldots + \alpha_{ks}f_{ks})) + \lambda_k f_k \\ &= \pi_j((\alpha_{ks} - \alpha_{k1})u_{k1} + (\alpha_{ks} - \alpha_{k2})u_{k2} + \ldots + (\alpha_{ks} - \alpha_{k,s-1})u_{k,s-1}) \\ &\quad +\lambda_k f_k \end{aligned}$$

The total transmission rental is therefore the loss rental

$$\pi_j((\alpha_{ks} - \alpha_{k1})u_{k1} + (\alpha_{ks} - \alpha_{k2})u_{k2} + \ldots + (\alpha_{ks} - \alpha_{k,s-1})u_{k,s-1})$$

plus the thermal constraint rental

$$\lambda_k f_k.$$

4.2 Linear programming model

We now consider a general network model with piecewise linear losses, making the dispatch problem a linear program. Here we model flow in a line k as the difference $f_k - h_k$ of two nonnegative flow variables f_k and h_k. Each flow is made up of S loss segments, so

$$f = \sum_{s=1}^{S} f_s$$

and

$$h = \sum_{s=1}^{S} h_s.$$

The total flow out from any node i into the network is then

$$e_i^\top \sum_s G_s f_s - e_i^\top \sum_s H_s h_s$$

where each matrix G_s and H_s models the flow in segment s. Since losses are accrued at the receiving end of the line, G and H are generalized network matrices with components

$$(G_s)_{ik} = \begin{cases} 1, & \text{if link } k \text{ has direction from node } i \\ -1+\alpha_{ks}, & \text{if link } k \text{ has direction to node } i \\ 0, & \text{otherwise,} \end{cases}$$

$$(H_s)_{ik} = \begin{cases} 1-\alpha_{ks}, & \text{if link } k \text{ has direction from node } i \\ -1, & \text{if link } k \text{ has direction to node } i \\ 0, & \text{otherwise,} \end{cases}$$

where α_{ks} is the loss slope on section s of the piecewise linear loss function.

The dispatch problem now becomes the following linear program:

$$\begin{array}{llcccc} \text{LP:} & \text{maximize} & -\sum_i \sum_{j\in O(i)} c_j x_j - \sum_i \sum_{j\in R(i)} b_j y_j & & & \\ & \text{subject to} & e_i^\top \sum_s G_s f_s - e_i^\top \sum_s H_s h_s - \sum_{j\in O(i)} x_j & \leq & -d_i, & [\pi_i] \\ [\rho_s] & 0\leq & f_s & \leq & u_s, & [\mu_s] \\ [\sigma_s] & 0\leq & h_s & \leq & l_s, & [\lambda_s] \\ & & LC(\sum_s f_s - \sum_s h_s) & = & 0 & [\tau] \\ & & V(\sum_s f_s - \sum_s h_s) + Yy + Wx & \leq & b & [\nu] \\ & & (x,y) & \in & X. & \end{array}$$

This is essentially the linear program solved by SPD. The loop flow constraints

$$LC(\sum_s f_s - \sum_s h_s) = 0$$

represent Kirchhoff's voltage law in the transmission network. In SPD these constraints are represented using voltage phase angles, from which an equivalent set of loop flow constraints can be obtained by summing voltage angle differences around oriented loops.

The transmission rental is by definition $R_t = \sum_i \pi_i g_i(f)$ which in this context gives

$$R_t = -\pi^\top \sum_s G_s f_s + \pi^\top \sum_s H_s h_s$$

But we have dual feasibility

$$\begin{array}{rcl} \nu^\top V + \tau^\top LC + \mu_s^\top - \rho_s^\top + \pi^\top G_s & = & 0, \quad s = 1,2,...,S \\ -\nu^\top V - \tau^\top LC + \lambda_s^\top - \sigma_s^\top - \pi^\top H_s & = & 0, \quad s = 1,2,...,S \end{array}$$

This means that

$$\begin{aligned} R_t &= -\pi^\top \sum_s G_s f_s + \pi^\top \sum_s H_s h_s \\ &= \sum_s (\nu^\top V + \tau^\top LC + \mu_s^\top - \rho_s^\top) f_s \\ &\quad + \sum_s (-\nu^\top V - \tau^\top LC + \lambda_s^\top - \sigma_s^\top) h_s \\ &= \nu^\top V(\sum_s f_s - \sum_s h_s) + \tau^\top LC(\sum_s f_s - \sum_s h_s) \\ &\quad + \sum_s (\mu_s^\top - \rho_s^\top) f_s + \sum_s (\lambda_s^\top - \sigma_s^\top) h_s \\ &= \nu^\top V(\sum_s f_s - \sum_s h_s) + \sum_s (\mu_s^\top - \rho_s^\top) f_s + \sum_s (\lambda_s^\top - \sigma_s^\top) h_s \end{aligned}$$

Now if Y and W are zero matrices, so that no constraints contain both flow and dispatch variables then at optimality we have

$$V(\sum_s f_s^* - \sum_s h_s^*) = b$$

and so the total transmission rental is

$$R_t = \nu b + \sum_s \mu_s^\top f_s^* + \sum_s \lambda_s^\top h_s^*.$$

Here the constraint including V could represent a security constraint for example. If Y or W are nonzero, then there are mixed constraints so

$$R_t = \nu(b - Yy^* - Wx^*) + \sum_s \mu_s^\top f_s^* + \sum_s \lambda_s^\top h_s^*.$$

Observe that for any line we will have either $f_{sk}^* = 0$ or $h_{sk}^* = 0$. Suppose $h_{sk}^* = 0$, and consider the term $\sum_s \mu_{sk} f_{sk}^*$. Recall

$$f_k^* = \sum_{s=1}^{S} f_{sk}^*$$

and suppose for some $s \leq S$, that

$$f_k^* = u_{k1} + u_{k2} + \ldots u_{k,s-1} + f_{ks},$$

where

$$f_{ks} < u_{ks}.$$

Dual feasibility for line k then gives

$$\begin{aligned}
\left(\nu^\top V\right)_k + \left(\tau^\top LC\right)_k + \mu_{k1} + \pi_i - \pi_j(1-\alpha_{k1}) &= 0 \\
\left(\nu^\top V\right)_k + \left(\tau^\top LC\right)_k + \mu_{k2} + \pi_i - \pi_j(1-\alpha_{k2}) &= 0 \\
\ldots &= 0 \\
\left(\nu^\top V\right)_k + \left(\tau^\top LC\right)_k + \mu_{ks} + \pi_i - \pi_j(1-\alpha_{ks}) &= 0
\end{aligned}$$

Now $f_{ks} < u_{ks}$ implies $\mu_{ks} = 0$, and so the last equation yields

$$\left(\nu^\top V\right)_k + \left(\tau^\top LC\right)_k = \pi_j(1-\alpha_{ks}) - \pi_i$$

We can then substitute to obtain

$$\begin{aligned}
\pi_j(1-\alpha_{ks}) - \pi_i + \mu_{k1} + \pi_i - \pi_j(1-\alpha_{k1}) &= 0 \\
\pi_j(1-\alpha_{ks}) - \pi_i + \mu_{k2} + \pi_i - \pi_j(1-\alpha_{k2}) &= 0 \\
\ldots &= 0 \\
\pi_j(1-\alpha_{ks}) - \pi_i + \mu_{k,s-1} + \pi_i - \pi_j(1-\alpha_{k,s-1}) &= 0
\end{aligned}$$

yielding

$$\mu_{kt} = \pi_j(\alpha_{ks} - \alpha_{kt}),\ t = 1, 2, \ldots, s. \tag{5}$$

In subsection 4.1 we derived the contribution of link k to R_t

$$R_t(k) = \pi_j\{(\alpha_{ks} - \alpha_{k1})u_{k1} + (\alpha_{ks} - \alpha_{k2})u_{k2} + \ldots + (\alpha_{ks} - \alpha_{k,s-1})u_{k,s-1}\}$$

for the case *without any other constraints on* f (i.e. when $V = 0$). This is consistent with

$$R_t(k) = \mu_{k1}u_{k1} + \mu_{k2}u_{k2} + \ldots + \mu_{k,s-1}\,u_{k,s-1}$$

because of (5).

What happens when $V \neq 0$? Now there are other constraints on f. In this case the dual variables π do not satisfy

$$\pi_i = \pi_j(1-\alpha_{ks})$$

and so the derivation of $R_t(k)$ in subsection 4.1 is not valid. Observe however that (5) implies that

$$\pi_j\{(\alpha_{ks} - \alpha_{k1})u_{k1} + (\alpha_{ks} - \alpha_{k2})u_{k2} + \ldots + (\alpha_{ks} - \alpha_{k,s-1})u_{k,s-1}\} = \sum_s \mu_{ks} f^*_{ks}$$

so it is still true that

$$\begin{aligned} R_t(k) &= \pi_j\{(\alpha_{ks} - \alpha_{k1})u_{k1} + (\alpha_{ks} - \alpha_{k2})u_{k2} + \ldots + (\alpha_{ks} - \alpha_{k,s-1})u_{k,s-1}\} \\ &= \sum_s \mu_{ks} f^*_{ks}. \end{aligned}$$

The total transmission rent (as defined to be $\sum_i \pi_i g_i(f)$) is therefore

$$R_t = \nu(b - Yy^* - Wx^*) + \sum_s \mu_s^\top f_s^*.$$

which includes a component due to the price ν on the constraints involving V, and the loss rental $\sum_s \mu_s^\top f_s^*$.

5 Distribution factors and losses

We now turn our attention to networks with a meshed topology and investigate the modelling of distribution factors when there are losses. Consider the set of constraints defined by

$$\begin{array}{rrcl} \text{SF:} & -e_i^\top Af & = & q_i, \quad i = 1, 2, \ldots, n, \\ & f & \in & F. \end{array}$$

Now suppose the set F is defined to be

$$F = \{f \mid f_k = \frac{\theta_i - \theta_j}{C_k}, \quad -K_k \leq f_k \leq K_k, \, \theta \in \mathbb{R}^n, \, \theta_1 = 0\},$$

where K_k is the thermal limit of the line k (from node i to node j), C_k the line's reactance, and θ_j is the voltage phase angle at node j. This models the loop-flow constraints in which we assign node 1 to be the *swing* bus. If we let C be a diagonal matrix of line reactances then we have

$$f = C^{-1}A^\top\theta$$

5.1 Lossless case

Without losses we obtain

$$\begin{array}{rcl} q & = & -Af \\ q & = & -AC^{-1}A^\top\theta \end{array}$$

Now to correspond with the choice of swing bus let $\bar{A}$ be A with its first row removed, and define $\bar{q}$ and $\bar{\theta}$ similarly. Then the $(n-1) \times (n-1)$ matrix $\bar{A}C^{-1}\bar{A}^\top$ has rank $n-1$, and so it is invertible. We can then write

$$\bar{\theta} = -\left(\bar{A}C^{-1}\bar{A}^\top\right)^{-1}\bar{q}$$

Thus

$$f = -\bar{C}^{-1}\bar{A}^\top\left(\bar{A}C^{-1}\bar{A}^\top\right)^{-1}\bar{q}. \tag{6}$$

The components of the matrix

$$D = \bar{C}^{-1}\bar{A}^\top\left(\bar{A}C^{-1}\bar{A}^\top\right)^{-1}$$

define *distribution factors* d_{ki} that give the flow on arc k that would result from an *injection* of one MW at node i and a withdrawal of one MW at node 1 (the swing bus). These flows are uniquely determined by i and the choice of swing bus. Observe that the formula (6) uniquely defines the flow in the lines for any injection pattern $-q$ with $\sum_{i=1}^{n} q_i = 0$. This flow pattern does not depend on the choice of swing bus.

Now consider the case with line losses. Here we have

$$\begin{array}{rlcl} \text{SF:} & g_i(f) - z_i & = & q_i, \quad i = 1, 2, \ldots, n, \\ & z_i & \geq & 0, \quad i = 1, 2, \ldots, n, \\ & f & \in & F. \end{array}$$

With no free disposal we get

$$\begin{aligned} q &= g(f) \\ q &= g(C^{-1}A^\top \theta) \end{aligned}$$

This gives a relationship between the voltage angles and the vector q of offtakes. In the absence of losses we have $q = -Af$ so any feasible solution must require $\sum_{i=1}^{n} e_i^\top q = 0$, because $\sum_{i=1}^{n} e_i^\top A = 0$. In other words flow is conserved. When there are nonzero losses we get

$$\sum_{i=1}^{n} e_i^\top q = \sum_{i=1}^{n} g_i(C^{-1}A^\top \theta) < 0$$

This does not enable a neat decoupling of flows into distribution factors as in the lossless case. Indeed the distribution factors vary with the dispatch.

5.2 Fixed linear losses

To get some intuition, suppose there are linear losses with slope α_k at the end of each line k. Recall generalized network matrices G and H with components

$$(G_s)_{ik} = \begin{cases} 1, & \text{if link } k \text{ has direction from node } i \\ -1 + \alpha_{ks}, & \text{if link } k \text{ has direction to node } i \\ 0, & \text{otherwise,} \end{cases}$$

$$(H_s)_{ik} = \begin{cases} 1 - \alpha_{ks}, & \text{if link } k \text{ has direction from node } i \\ -1, & \text{if link } k \text{ has direction to node } i \\ 0, & \text{otherwise,} \end{cases}$$

where α_{ks} is the loss slope on section s of the piecewise linear loss function. We have

$$q = -Gf + Hh$$

Suppose that $h = 0$. In other words suppose that we can choose flow directions for the columns of G that turn out to be the actual flow directions in the dispatch. Then

$$q = -GC^{-1}A^{\top}\theta$$

Now, since $A^{\top}$ has rank $n-1$, the matrix $GC^{-1}A^{\top}$ is not invertible. If we choose to set $\theta_1 = 0$, and form $\bar{A}$ and $\bar{\theta}$ by deleting row 1, then we obtain

$$q = -GC^{-1}\bar{A}^{\top}\bar{\theta}$$

Here $D = GC^{-1}\bar{A}^{\top}$ is an $n \times (n-1)$ matrix, and so it is also not invertible.

Suppose we ignore q_1 and the first row of G. Then we obtain

$$\bar{q} = -\bar{G}C^{-1}\bar{A}^{\top}\bar{\theta}$$

$$\bar{\theta} = -(\bar{G}C^{-1}\bar{A}^{\top})^{-1}\bar{q}$$

and using $f = C^{-1}\bar{A}^{\top}\bar{\theta}$ we get

$$f = -C^{-1}\bar{A}^{\top}(\bar{G}C^{-1}\bar{A}^{\top})^{-1}\bar{q}$$

This defines a matrix of distribution factors

$$D = C^{-1}\bar{A}^{\top}(\bar{G}C^{-1}\bar{A}^{\top})^{-1}$$

giving

$$f = -D\bar{q}$$

The (unknown) value of offtake q_1 at the swing bus is then determined by

$$q_1 = -e_1^{\top}Gf = e_1^{\top}GD\bar{q}.$$

Now suppose that $f \neq 0$ and $h \neq 0$. This will be the general case when we do not know the flow directions in advance. Then we have

$$q = -Gf + Hh$$

and

$$f - h = C^{-1}A^{\top}\theta.$$

Now because $G \neq -H$, we cannot proceed to extract distribution factors.

This is a fundamental problem for analysing networks with losses, even if these are linear. The reason is that the flows are no longer linear functions of the injections. This is because the flow arriving at a node is not a linear function of what is sent. This is true if the flow direction is known a priori, but if it switches direction then there is a kink in this function at zero. With constant marginal losses, one might (as above) construct a different PTDF matrix G for all possible combinations of power flow direction, but there is not a single matrix that works for all injections.

5.3 Quadratic losses

We now return to the case of quadratic loss functions $g_i(f)$, where

$$q = g(f). \tag{7}$$

The distribution factors now vary with the dispatch. We can obtain a marginal result by differentiating (7) to obtain a Jacobian

$$J(f) = \frac{\partial(q_1, q_2, \ldots, q_n)}{\partial(f_1, f_2, \ldots, f_m)}$$

where

$$J_{ik}(f) = \frac{\partial g_i(f)}{\partial f_k}.$$

Then at a given dispatch solution

$$\Delta q = J(f)\Delta f$$

In the case of symmetric quadratic losses

$$g_i(f) = \sum_{k \in \mathcal{F}(i)} \left(-f_k - \frac{r_k}{2} f_k^2\right) + \sum_{k \in \mathcal{T}(i)} \left(f_k - \frac{r_k}{2} f_k^2\right)$$

so

$$J_{ik}(f) = \begin{cases} -1 - r_k f_k, & k \in \mathcal{F}(i) \\ 1 - r_k f_k, & k \in \mathcal{T}(i) \\ 0, & \text{otherwise} \end{cases}.$$

At a given operating point

$$\Delta q = J(f)\Delta f$$

corresponds to

$$\Delta q = -G(f)\Delta f$$

where

$$\begin{aligned} G(f) &= -J(f) \\ &= \begin{cases} 1 + r_k f_k, & k \in \mathcal{F}(i) \\ -1 + r_k f_k, & k \in \mathcal{T}(i) \\ 0, & \text{otherwise} \end{cases} . \end{aligned}$$

We can eliminate a swing bus (node 1, say) and compute a matrix of marginal distribution factors

$$D(f) = C^{-1}\bar{A}^\top(\bar{G}(f)C^{-1}\bar{A}^\top)^{-1}$$

which satisfy

$$\Delta f = -D(f)\Delta\bar{q}.$$

The marginal change in offtake Δq_1 at the swing bus is then determined by

$$\Delta q_1 = -e_1^\top G(f)\Delta f = e_1^\top G(f)D(f)\Delta\bar{q}.$$

It is important to make several observations:

1. The marginal distribution factors vary with the flows f which depend on the dispatch. They are not constant factors.

2. The marginal distribution factors do not give the proportion of flow from a point-to-point flow that is carried by line k.

3. For a given offtake q_i^*, it is possible to compute the total flow from a point-to-point flow that is actually carried by line k by computing

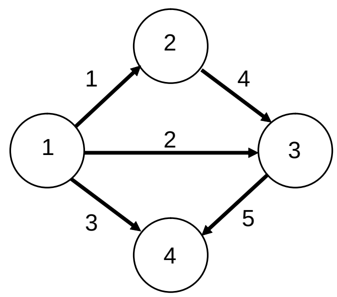
$$f_k^* = -\int_0^{h^*} D_{ik}(f(q))dq. \tag{8}$$

4. In principle, the computation in (8) requires computation of a flow $f(q)$ for every offtake in the interval $[0, q^*]$. It would be easier to obtain f_k^* directly by one of these computations. This is done in example 5 in the next section, which compares the distribution factors estimated from a lossless model with the distribution factors computed for quadratic losses.

6 Numerical Examples

6.1 Example 1:

Consider the network shown in the figure below.



Example network showing labelling of nodes and lines

This has incidence matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

and suppose the reactances are

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here the reduced node-arc incidence matrix is

$$\bar{A} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

$$\bar{A}^{\top} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Then the matrix of distribution factors is

$$
\begin{aligned}
D &= C^{-1}\bar{A}^{\top}(\bar{A}C^{-1}\bar{A}^{\top})^{-1} \\
&= \begin{bmatrix} -0.625 & -0.25 & -0.125 \\ -0.25 & -0.5 & -0.25 \\ -0.125 & -0.25 & -0.625 \\ 0.375 & -0.25 & -0.125 \\ 0.125 & 0.25 & -0.375 \end{bmatrix}
\end{aligned}
$$

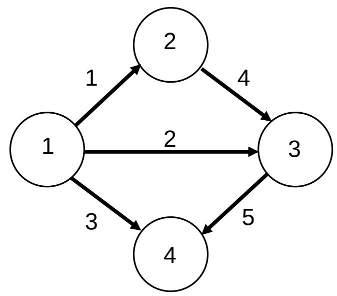
We can check this, by injecting -1 at each of nodes 2,3, and 4. This gives flows in the lines of

$$
\begin{aligned}
f &= -Dq \\
&= \begin{bmatrix} -0.625 & -0.25 & -0.125 \\ -0.25 & -0.5 & -0.25 \\ -0.125 & -0.25 & -0.625 \\ 0.375 & -0.25 & -0.125 \\ 0.125 & 0.25 & -0.375 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

as expected.

6.2 Example 2

We now consider the same network but set the reactance of line 2 to be 2, so this has a greater impedance. The network is:



Example network showing labelling of nodes and lines

This has incidence matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} D &= C^{-1}\bar{A}^{\top}(\bar{A}C^{-1}\bar{A}^{\top})^{-1} \\ &= \begin{bmatrix} -0.666\,67 & -0.333\,33 & -0.166\,67 \\ -0.166\,67 & -0.333\,33 & -0.166\,67 \\ -0.166\,67 & -0.333\,33 & -0.666\,67 \\ 0.333\,33 & -0.333\,33 & -0.166\,67 \\ 0.166\,67 & 0.333\,33 & -0.333\,33 \end{bmatrix} \end{aligned}$$

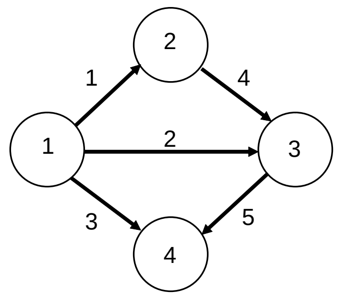
Checking

$$
\begin{aligned}
f &= -Dq \\
&= \begin{bmatrix} -0.666\,67 & -0.333\,33 & -0.166\,67 \\ -0.166\,67 & -0.333\,33 & -0.166\,67 \\ -0.166\,67 & -0.333\,33 & -0.666\,67 \\ 0.333\,33 & -0.333\,33 & -0.166\,67 \\ 0.166\,67 & 0.333\,33 & -0.333\,33 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 1.\,166\,7 \\ 0.666\,67 \\ 1.\,166\,7 \\ 0.166\,67 \\ -0.166\,67 \end{bmatrix}
\end{aligned}
$$

as expected.

6.3 Example 3

We now consider the effect of quadratic losses on distribution factors. Consider the network shown in the figure below.



Example network showing labelling of nodes and lines

This has incidence matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now suppose we have symmetric quadratic losses defined by r_k, where

$$r^\top = \begin{bmatrix} 0.02 & 0.04 & 0.02 & 0.04 & 0.02 \end{bmatrix}$$

We will compute the arc flows f when $\bar{q} = \begin{bmatrix} -0.98 \\ -0.97 \\ -0.96 \end{bmatrix}$. This involves finding a solution to

$$g_i(f) = \sum_{k \in \mathcal{F}(i)} \left(-f_k - \frac{r_k}{2} f_k^2\right) + \sum_{k \in \mathcal{T}(i)} \left(f_k - \frac{r_k}{2} f_k^2\right)$$

$$\begin{aligned}
-f_4 - 0.02f_4^2 + f_1 - 0.01f_1^2 &= 0.98 \\
f_2 - 0.02f_2^2 + f_4 - 0.02f_4^2 - f_5 - 0.01f_5^2 &= 0.97 \\
+f_3 - 0.01f_3^2 + f_5 - 0.01f_5^2 &= 0.96 \\
f_1 + f_4 - 2f_2 &= 0 \\
2f_2 + f_5 - f_3 &= 0
\end{aligned}$$

Solution is:

$f_1 = 1.1512$
$f_2 = 0.65429$
$f_3 = 1.1409$
$f_4 = 0.15741$
$f_5 = -0.16764$
$r_1 = 0.02$
$r_2 = 0.04$
$r_3 = 0.02$
$r_4 = 0.04$
$r_5 = 0.02$

We can compare this with the solution from example 2 that assumed no losses. This is

$$f = \begin{bmatrix} 1.1667 \\ 0.66667 \\ 1.1667 \\ 0.16667 \\ -0.16667 \end{bmatrix}$$

We can also compare this with an approximation that uses marginal losses to compute distribution factors. This gives

$$G_{ik}(f) = \begin{cases} 1 + r_k f_k, & k \in \mathcal{F}(i) \\ -1 + r_k f_k, & k \in \mathcal{T}(i) \\ 0, & \text{otherwise} \end{cases} .$$

$$G = \begin{bmatrix} 1 + r_1 f_1 & 1 + r_2 f_2 & 1 + r_3 f_3 & 0 & 0 \\ -1 + r_1 f_1 & 0 & 0 & 1 + r_4 f_4 & 0 \\ 0 & -1 + r_2 f_2 & 0 & -1 + r_4 f_4 & 1 + r_5 f_5 \\ 0 & 0 & -1 + r_3 f_3 & 0 & -1 + r_5 f_5 \end{bmatrix}$$

$$G = \begin{bmatrix} 1.023 & 1.0262 & 1.0228 & 0 & 0 \\ -0.97698 & 0 & 0 & 1.0063 & 0 \\ 0 & -0.97383 & 0 & -0.9937 & 0.99665 \\ 0 & 0 & -0.97718 & 0 & -1.0034 \end{bmatrix}$$

$$S = \begin{bmatrix} -0.97698 & 0 & 0 & 1.0063 & 0 \\ 0 & -0.97383 & 0 & -0.9937 & 0.99665 \\ 0 & 0 & -0.97718 & 0 & -1.0034 \end{bmatrix}$$

$$W = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$D = C^{-1}W(SC^{-1}W)^{-1} = \begin{bmatrix} -0.67737 & -0.34560 & -0.17391 \\ -0.17064 & -0.34057 & -0.17138 \\ -0.17290 & -0.34507 & -0.67855 \\ 0.3361 & -0.33553 & -0.16884 \\ 0.16838 & 0.33606 & -0.33580 \end{bmatrix}$$

$$\begin{aligned} f &= -Dh \\ &= \begin{bmatrix} -0.67737 & -0.34560 & -0.17391 \\ -0.17064 & -0.34057 & -0.17138 \\ -0.17290 & -0.34507 & -0.67855 \\ 0.3361 & -0.33553 & -0.16884 \\ 0.16838 & 0.33606 & -0.33580 \end{bmatrix} \begin{bmatrix} -0.98 \\ -0.97 \\ -0.96 \end{bmatrix} \\ &= \begin{bmatrix} 1.166 \\ 0.6621 \\ 1.1556 \\ 0.15817 \\ -0.16862 \end{bmatrix} \end{aligned}$$

We now compare with the true f which is
$f_1 = 1.1512$

$f_2 = 0.654\,29$
$f_3 = 1.140\,9$
$f_4 = 0.157\,41$
$f_5 = -0.167\,64$

This example shows that the use of approximate distribution factors for flows with losses needs some care to ensure that the approximations retain the accuracy required.

References

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